Introduction to Classical Mechanics

Assistant Professor Doctor Anurag Tripathi
Indian Institute of Technology, Hyderabad

Lecture-5

d’Alembert Principle.

Hello, last time in the video I introduced holonomic constraints. And I also talked about a specific example of 2 particles connected by a rigid rod. And we looked at the virtual work in that case. Just to remind you, we found that the virtual work done by the system was 0 but individual forces acting on each of the particles, so if you take particle number 1 and you look at the forces which are acting on it due to constraint. And if you find out the virtual work it is not 0, only for the entire system it turned out to be 0.

And that is generically true for constraint forces. It is not that the work done by the forces acting on individual particles is 0, it is only when you consider all the particles of the system and look at all the constraints, then all the and you add up the work done by these forces on each particle and you sum them up, that is 0. Okay, so that is what one thing which should we should remember and that is what we are going to utilize today.

Now, goal for today's lecture is to write down the principle which is called d'Alembert principle. This will be our starting point for writing down equations of motion later. So, that is what we are going to do and the task is quite simple here and the principle is very nice and simple. So, let us begin with that. So, imagine we have a system of n particles. So, a lot of particles and each particle is experiencing some forces due to constraint and also forces which are not due to constraint.

So, I divide all the forces on each particle in 2 sets, f primes and fs, primes with are due to constraint and f without primes are not due to constraint. And then just now I told you that if I take the virtual work done by all of these forces f primes and you add all these virtual works, then it will turn out to be 0. And that is what we are going to utilize in writing down d’Alambert principle.

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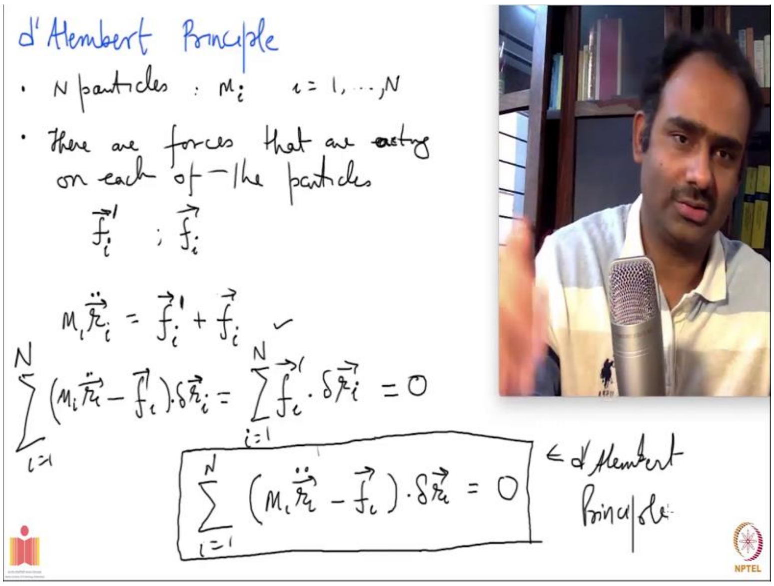

So, here we are. So, I am looking at a system of n particles, each of mass m i, and where i runs from 1 to N, the N number of particles. I am writing down small m right now. And then there are forces of constraint that are given by the equations, the holonomic constraints. So, let us say there are k number of constraints. So, phi, 1 phi 2 or so on and so forth to phi k. And these being equal to 0 are the equations of constraints, so that is what I am imagining right now.

Let me write down, there are forces of not of, let us say there are forces that are acting on the particle, that are acting on each of the particles. And as I said I will divide them into 2 categories, one is f prime. So, particle number i has a force f i prime acting on it, and this fi prime is due to constraints, and this is the sum of all constraint forces, okay. If there are several forces which are acting due to constraint, I add them all of all of them, and that is what f prime is.

And another set is f, and this is also the sum of all forces other than the constraint forces, okay, on particular number i. Now, each of the particle is going to evolve in time, according to Newton's second law. So, if I write down mass into acceleration, I would double dot here. And that will be the sum of f prime and f, f prime i plus f. Right? That is correct. Now, you see, we are not very happy and excited about this equation. This is true.

This equation is true, but it is not very nice, because you see on the right-hand side, you have a f prime so you need to know what the force of constraint is in order to solve this equation and tell what the trajectory of the particle is. But now you may not necessarily know what are the forces of constraint acting. Let us say, let us take an example to understand this. Let us imagine that you have one surface, let us say a flat surface and you have a particle which is constrained to remain on that surface.

And on the top of it, you have certain very massive particles, which very strongly exert forces on this guy. So, let us say you have a surface, a particle on it, and then there are other particles here and there which are very massive, and they are all moving around because they are all interacting with each other. This guy will also move around because it is getting forces from all other particles, but this one has to remain on the surface.

Now, it may naively appear that I can plug all this and in this equation a Newton's second law, which I have written down here, okay and solve it. But the problem is, the force which the surface is going to exert on the particle here. So, let us say this is the. Let us say this is your surface. And your particle is going to be here on this. Now, if the particles which are at other places, the massive, very massive ones, if they are exerting a very strong force downwards, let us say the force is very, very strong.

Then for the particle on the surface to remain on the surface, the surface has to push it strongly upwards. But if let us say these particles move away in time, they go further away. So, that the full of these particles on the guy here is lesser, much smaller let us say. Then the reaction force by the surface will also be much smaller, meaning f prime will be smaller in magnitude. So, the forces f primes are not known a priori, they evolve as the system evolve.

And that is why this equation though correct, it is not very useful in solving your problem, unless you could write down explicitly what f i primes are. So, that is the, that is the difficulty we face with this way the equations of motion are written down right now in front of us. And that is what the situation is. And that is what we want to address in the remaining of the lecture. So, let us see what we can do about this.

What I will do is I will take the f i, the forces which are other than the constraint forces to the left-hand side and let us see what we get mi, ri double dot minus f i equals fi prime. I wish I had written it more neatly but Okay, let me just, equals fi prime. Okay, trivial. Now, what I do is I take my system and do virtual displacements that are consistent with the forces of constraint at that time. So, let me do what full displacements and dot both the left- and right-hand sides with virtual displacements.

This might already ring a bell to you. Now, as I said a few minutes ago, that the right-hand side is not 0, that the virtual work done by each, virtual work done on each particle by these forces constant is not necessarily 0. You have seen this especially in the case of 2 particles which are rigidly attached. And but if I sum over the entire system, okay, then this is true. And that is why I will sum over all the particles i equals 1 to N.

Now, this is true that the virtual work done is 0 and this is what is d'Alambert 's principle. It says that if I sum over all the particles, I am just writing down the equation again, mi ri double dot minus f i dotted with the virtual displacements, this is equal to 0. Let me put it in a box. Nice, you might have already realized that this is a nice equation, because your f primes are gone now, they are not hearing the principal, the problem. And this is d’Alambert Principle.

Okay, pay attention to this. Let us see what it is saying. It is saying that mi ri double dot minus f i dotted with delta ri is 0. Now, imagine for a moment that there were no forces of constraint. If that was the case then all the Delta ri's the virtual displacements of each of the particles, you could have chosen them all independently of each other, they will be all independent of each other.

See, the Delta ri's are constrained because of the constraint equations. You remember some time back we wrote down d phi and then we wrote down d phi, which was in terms of delta ri's and that was the equation of constraint, that was the constraint that virtual displacements had to satisfy. Let me go back, let me try to see if I can find that equation.

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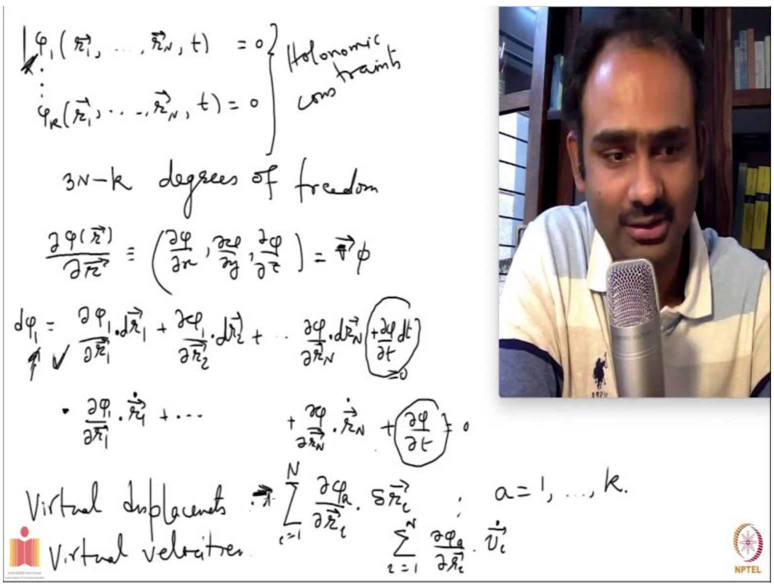

Yeah, here for example, here if you see, here, this one. You see the virtual displacement delta ri had to satisfy this equation, they will not, so he you cannot independently choose one delta r1 to be this, delta r2 to be to this delta r3 to be this, have to satisfy the constraints. And that is what I am saying here.

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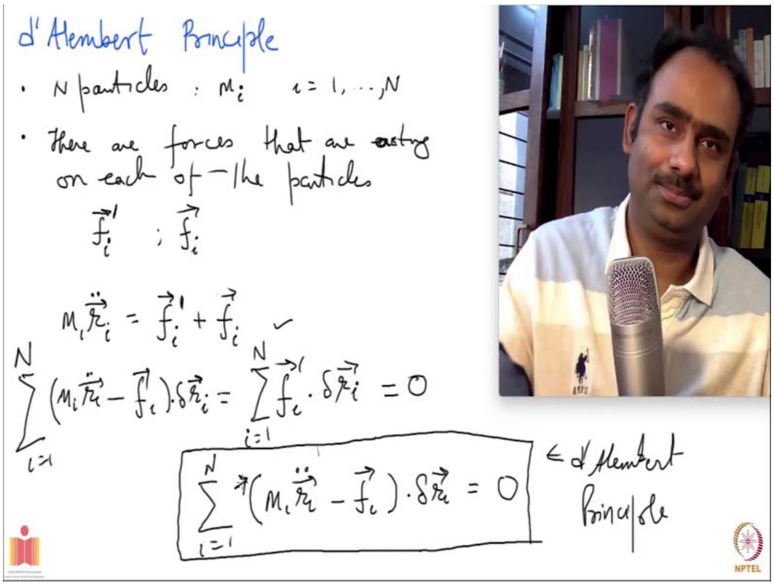

But imagine if your system was not constrained by any constraint forces, then you could choose delta ri's to be independent of each other. And then in that case, for this sum to be 0, you will conclude that whatever is written in the square brackets here, mri, mi ri double dot minus fi, they all have to individually vanish, just because a Delta ri's are independent of each other. So, if that was the case, then you recover your Newton's equations, which is just mass times acceleration is the force and there is only one force because the constraint is gone.

But now the situation is that we have constraints. But then the delta is are not independent of each other, so I cannot conclude that m i ri double dot minus f is 0, which is good that otherwise it would mean that I have made a mistake. But then it also suggests that if I could write down this equation, I take this equation and instead of using Delta, instead of using ri's, I use the generalized coordinates which I was talking about some time back, remember generalized coordinates are all independent of each other.

Now, if I could write down this relation using generalized coordinates, then the delta ri's will get replaced by delta qi's and because they are independent of each other, the displacements delta qi's could be taken independently of each other. And then whatever would be left in this square bracket here, not square, this is a round bracket here, this is a round bracket, this bracket is called round bracket. That I would be able to say that that part is 0, just like I was saying here for the case of a system in which there are no forces of constraint.

And that would give me my equations of motion without the forces of constraint, and that would be our achievement. And that is what we are after, writing down equations of motion without constraint forces. But the price to pay would be that we will be using generalized coordinates instead of Cartesian coordinates.

But that is not a big price, that is okay, we are quite comfortable with using generalized coordinates. So, that is the goal for next video. And those equations are which we are after, they are called Euler Lagrange equations and that is what we are going to obtain next time.