**Introduction to Classical Mechanics**
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**Lecture 49**
**First Integral of a Heavy Symmetric Top**

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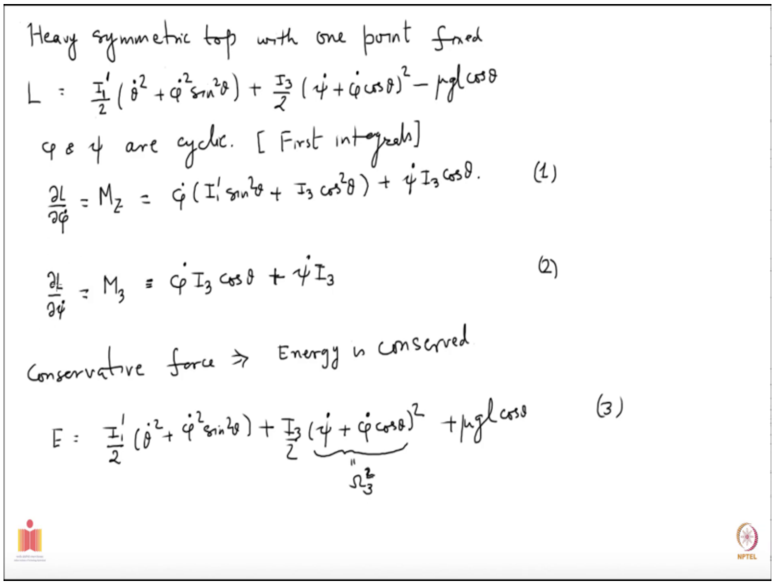


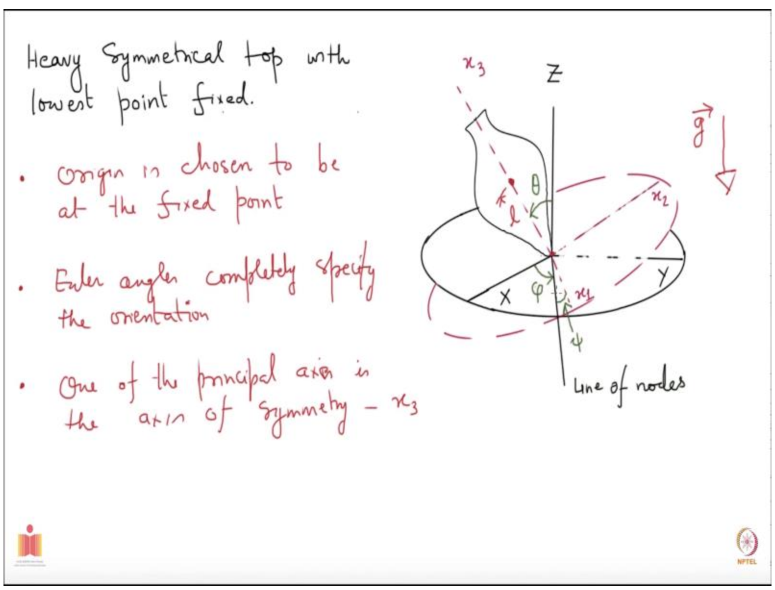


So, let us continue our discussion of Heavy Symmetric Top with one point fixed that is what we were looking at last time. Where that point is the lowest point of the top. So, the Lagrangian we wrote down last time was this the Lagrangian of the system is I1 prime over 2 and then you have theta dot square plus phi dot square sin square theta, these are your euler angles and their derivatives and you have then I3 over 2 psi dot plus phi dot cos of theta square minus the potential energy term which is mu gl cos of theta, where mu denotes a total mass of this top.

So, we quickly note that the coordinates phi and psi are cyclic you see phi and psi only appear as derivatives not otherwise. So, our coordinates phi and psi are cyclic and you know that they imply conservation of their corresponding conjugate momentum, which means that p phi which is del L over del psi del phi dot which is p phi, which is also same as let us go back and see what is phi dot, see phi remember phi was the first rotation we talked about and it happens about the Z axis.

So, this corresponds to the angular momentum Mz. So, you take your angular momentum and project it along Z that is what is del L over del phi dot and this comes out to be what? Del L over phi dot, so this is and you are not going to contribute, so this term will contribute and this term will contribute.

So, if I take the derivatives I get this, phi dot I1 prime sin square theta, so there is the first this contribution from here that 2 cancels with this 2 here plus I3 cos square theta that is from the square term and then from the cross term we get psi dot I3 cos of theta that is let us say our equation 1.

Now, psi is also cyclic. So, I write down del L over del psi dot will be a constant. So, here Mz is a constant, now del L over del psi dot will be constant and what is del L over del psi dot let us go back, psi dot. So, this rotation psi was happening about the x3 axis remember that the final rotation. It was about the final location of your how do you say anyway, it was about x3 axis. This was the final rotation. So, I call it the M3. So, we remember we are denoting by 1, 2 and 3 these coordinates x1, x2 and x3 and by capital X, Y and Z the fixed coordinates and that is why I am putting a 3.

So, here I have used the fact that the conjugate momentum corresponding to this rotation is angular momentum. So, that us your M3. Let us, take a derivative with respect to psi dot. So, these two do not contribute. This does not contribute only this term contributes and of course, there are two terms one from psi dot square and another from the cross term and you get phi dot I3 cos of theta plus psy dot I3.

Let us, call it equation number 2. So, these are our two first integrals we get. We also have one first integral because this is a conservative system and total energy will be conserved. So, we have one more, so conservation of energy conservative force implies energy conservation total energy conservation.

So, one more first integral and that is the total energy E is conserved which we have already seen but I will write theta dot square plus phi dot square sin square theta these are your kinetic term which I am writing right now I3 over 2 psy dot plus phi dot cos theta square this is your kinetic and then you have the potential energy mu g l cos theta, that is our equation number 3.

Now, look at this part, this is, you remember what this was? This was the kinetic energy was half I1 prime omega 1 square plus then you had I2 prime omega 2 square and then the third was half I3 omega 3 square. So, if you recall, this is just the expression of omega 3 square there is a reason why I am writing it like this here.

So, these are our first integrals which we can now look at because they are simpler than second order differential equations. So, what I can do immediately is look at this equation 1 and 2, they both involve phi dot and psy dot there is no nothing else with the derivative. So, I can solve these two because my Mz and M3 are constants because they are conserved quantities. I can solve these two for phi dot and psy dot and that will be in terms of theta. So, let me write it down.

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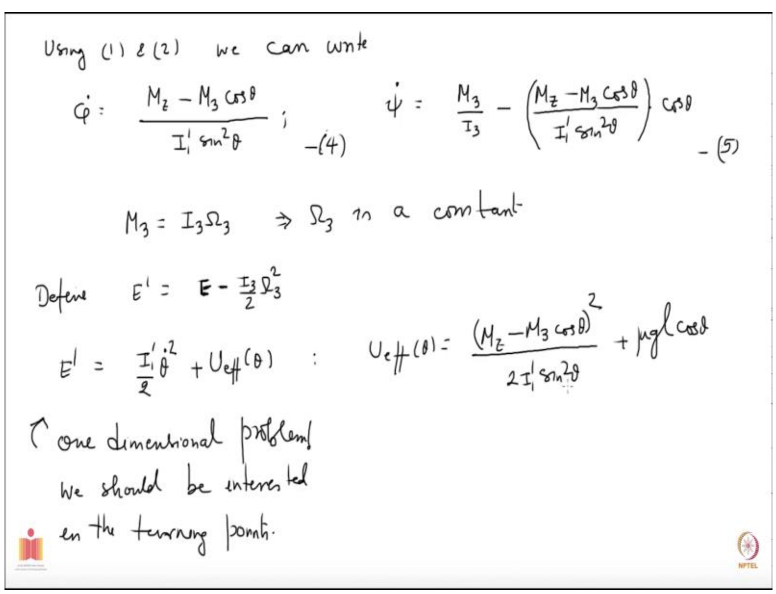

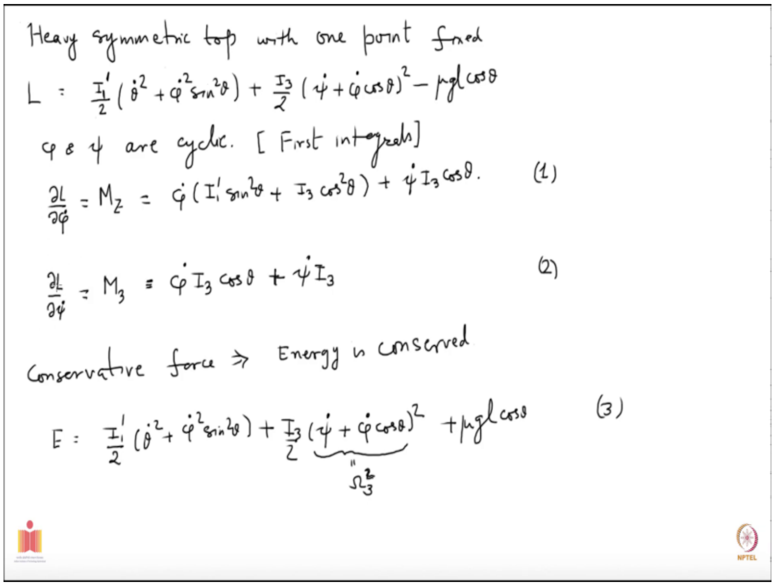


So, using 1 and 2 we can write phi dot is equal to Mz minus M3 cos of theta the whole thing divided by I1 prime sin square theta and then you will also have psy dot, maybe I should give a little bit more gap, sin square theta and your psy dot will be M3 over I3, this you can check and Mz minus M3 cos of theta over I1 prime sin square theta and then the entire thing is multiplied with the cos of theta, maybe I should put a bracket, this looks fine.

So, let me call this equation 3 and 4 this is 3, this is 4, no 4 and 5, equation 4 I think there is a way I can do it. So, this is 4 and this is 5 that is good. Now, note that your M3 is just I3 omega 3 third component of angular momentum is I3 omega 3 because these are your principal axis and because your M3 is a constant which we have seen here, M3 is a constant because psy is cyclic your omega 3 is also constant, that is good.

Now, what I do is let me go back to E. So, you have E equal to this thing plus half I3 omega 3 square, now omega 3 is constant this is constant, so I take this entire piece to the left and combined with E to define a new quantity, which I call E prime. So, I define E prime is equal to E minus I3 by 2 omega 3 square. It makes sense to define this because these are constants, E is constant that is the total energy and these entire quantity is a constant.

So, with this I have now E prime is equal to what? Let us go back this term plus that term. So, this term gives you a theta dot square term, plus something which involves only theta and something involves only theta and in this phi dot I am going to substitute this thing, because I have already determined phi dot in terms of theta.

So, I get when I substitute in E prime the following, I get I1 prime by 2 theta dot square that is this thing and then phi dot is expressed in terms of theta this is any way involving theta, this is gone to the left side, this involves only theta. So I define a new quantity, which is a function of only theta and I call it U effective theta, where U effective theta is defined by Mz minus M3 cos of theta this whole thing is squared and we should divide it by 2 I1 prime sin square theta make sure that there are no mistakes I think this is correct mu g l cos of theta, that is nice.

And it is nice because it looks very familiar, does it look familiar this equation. Well you have encountered the same thing when we were looking at a two body problem. Remember the equation for R coordinate. The radial separation. It was E equals half mu R dot square plus U effective R. So, instead of R you have theta here, but as far as the mathematics is concerned and other logics is concerned is the same there is no difference.

So, we can do what we did there. So, we are happy. We have a one dimensional problem as far as it theta variable is concerned. So, what we would like to know here is, let us recall what we were asking when we had this equations in the capital problem or two body problem. We wanted to know where the turning points are, where theta dot goes to 0 and depending on the turning points we knew where between what limits are the particle is going to move.

So, similarly here once we have found the turning points we will know that the theta has to very within those limits of theta which we determined by finding the turning points. So, that is what we will do. So, let me just write out. We should be interested in now in the turning points that is good. But before we do all that, let us do a little bit more of work. Now, I will make a substitution I will define u as a new variable and that will be equal to cos of theta.

So, what I am going to do is I am going to take my E prime which is here and wherever I have cos theta I am going to substitute u, see this will become u, this will become u, this will become 1 minus u square. So, as far as U effective is concerned you will immediately have it in terms of u and theta dot square will also give you a nice expression of u which I will show.

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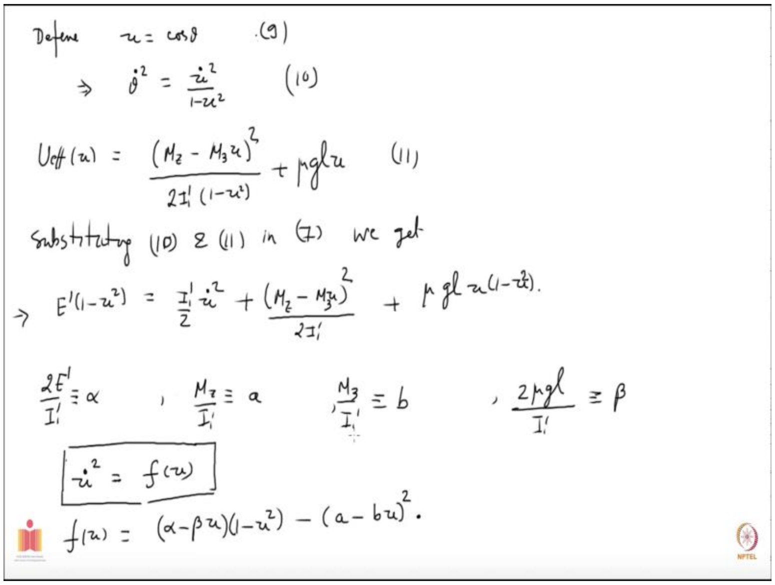


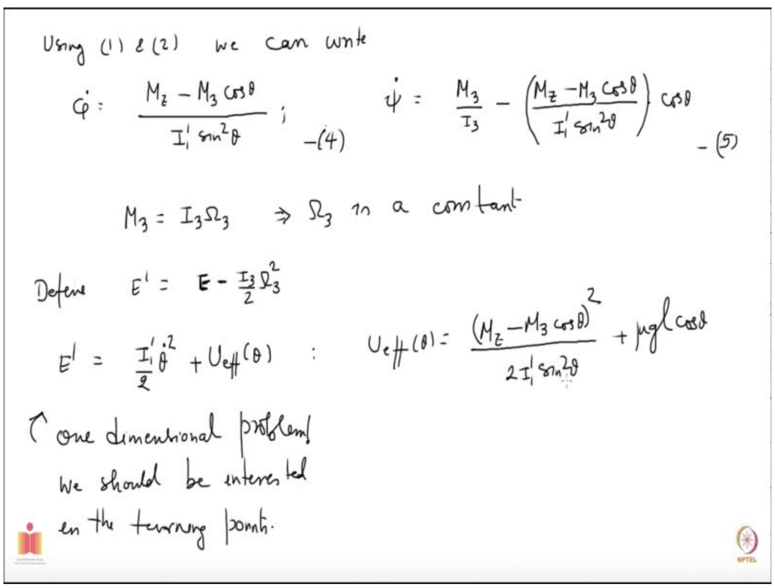


So, let us define u equals cos of theta, which gives you immediately u dot, so maybe I can write it is obvious so you can immediately get from this that u dot square all you have to do is just take a derivative and you get the following u dot square is not this way I want to theta dot square from this you get theta dot square as u dot square over 1 minus u square, that is nice.

And my U effective becomes U effective of u is Mz minus M3 u, Mz minus M3 u whole squared over 2 I1 prime 1 minus u square, u squared over 2 I1 prime 1 minus u square plus mu g l u, that is what our U effective would be and if you substitute this let us call it what 5, 6 and 7 maybe no.

Let us, call it 6, let us call this one 7 and this one 8 and 9, 10, 11. So, I am going to put 10 and 11 into 7 and 8 into 7.

So, what do I get? I get substituting, substituting 10 and 11 in 7, what do I get? Let us go back. So, you get the following E prime 1 minus u square, so I am taking 1 minus u square also and multiplying on the left hand side and this is equal to I1 prime over 2 u dot square plus Mz minus Mu square 2 I1 prime plus mu g l u times 1 minus u, please check that is correct. 1 minus u square I think, what was that? 1 minus u square that is because of this term getting multiplied 1 minus u square, that is correct. So, that is nice.

Now, what I will do is, I will take the coefficient of u dot square and divided this coefficient throughout this equation. So, I am going to divide everything by 2 I1 prime, I am going to divide everything by I1 prime by 2, so that the coefficient of this becomes 1 that is what I am going to do and for that I mean, of course I will I mean I do that and I also define.

So, you see this E prime will get divided by E prime will carry a factor with it 2 over I1 prime. Because will you divide it becomes 2 E1 prime over I1 prime, so 2 E1 prime over I1 prime then you will have this one will become 1, the 2 when I multiply this entire equation by 2 to get rid of this there will be a 2 here which will cancel this 2 and then you have I1 prime divided here.

So, that will make I1 prime square which I want to bring in this and it will be dividing Mz over I1 prime and M3, M3 over I1 prime, so I will have M3 over I1 prime and I will have, first let me write Mz, Mz over I1 prime, then I will have M3 over I1 prime and then the last term will have, what will it have it will have 2 will come in the numerator. So, 2 mu g l over I1 prime this will have this, that is correct.

So, I will define all these constants to be the following, this is very standard notation. This is alpha, this should have been called beta, this is called Mz over I1 prime is called a M3 over I1 prime is called b. So, that is these are what I am defining. So, with this my equation becomes the following, so you can substitute this this definition into this equation and you will obtain u dot square, u dot square and the entire remaining thing will be some function of u where fu is given by the following. It will be alpha minus beta u then you will have 1 minus u square that is correct then minus a minus b u square, that is what you will get.

We will continue with further discussion on this in the next video. So, please before watching this next video, please ensure that you have reproduce all these steps which are fairly easy and then we will be ready to analyse the motion of this top, which is quite interesting actually. See you then.