Introduction to Classical Mechanics
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Lecture No. 48
Lagrangian of a heavy symmetric top

Till now, we have studied Spherical Rocks freely moving space and also we discussed a symmetric rock which is freely moving space which are also called as Spherical Tops or Symmetric Tops. Now, we would like to discuss about a symmetric top that is moving around in uniform gravitational field for example on the surface of earth where the gravity is you can consider as constant over the dimensions of the top and you can imagine the one point of the top the lowest point it is, it is fixed at one point so that is what we want to consider but before we do that let us see that we have done last time.

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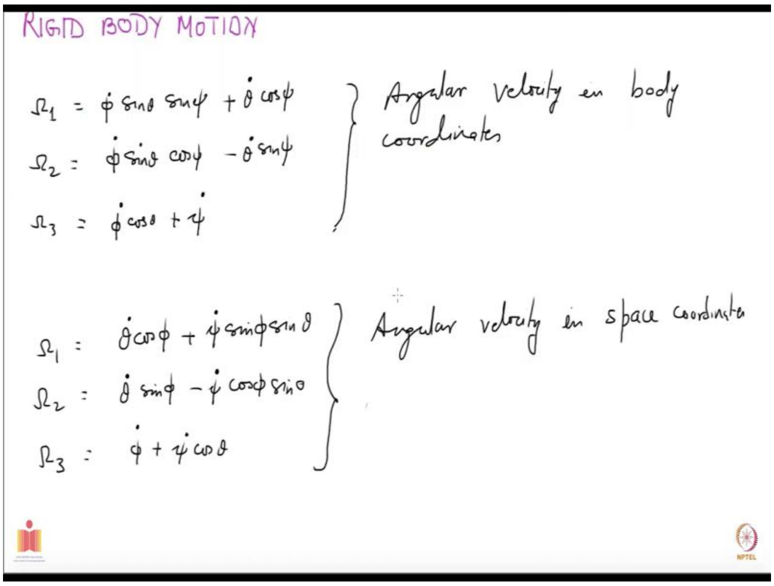

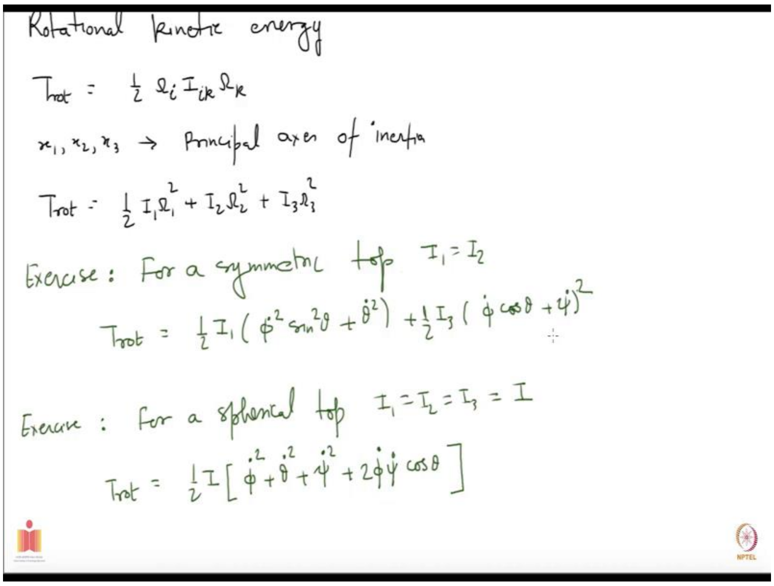

So, we wrote down angular velocity omega in body coordinates so, we had this expression in the first half of this page and then we also wrote down expressions for the kinetic energy due to rotation and then you had this omega I omega, this structure and we also wrote down this kinetic energy for a symmetric top and this is one thing which we are going to utilize later in this video today.

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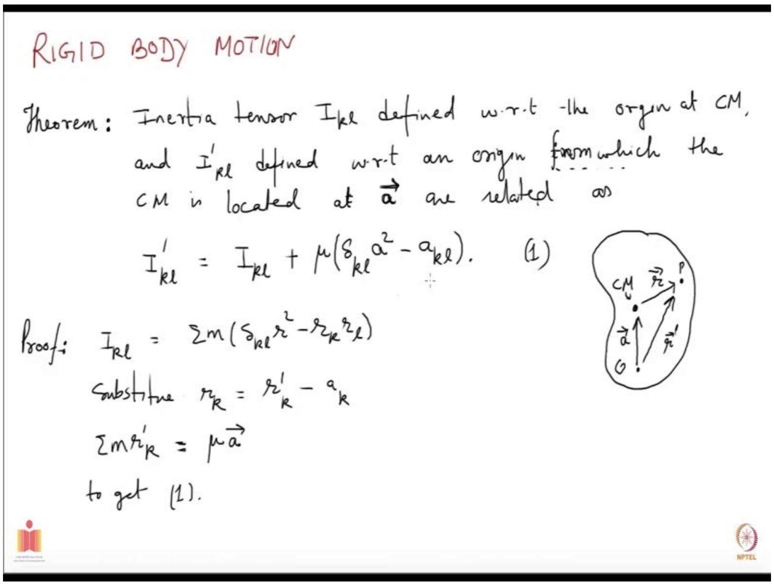

So, before I take up the top I want to write down two theorems which are fairly easy. So, theorem number 1 is the following and I will prove it or at least give as an exercise. So, we have been defining our inertia tensor with origin chosen as at the center of mass of the body axis that is what we did but let us say you are interested in for some reason finding out the inertia tensor with origin not origin not at the center of mass but at some other point then there is a relation and that is the theorem about.

So, the theorem says that the inertia tensor Ikl defined with respect to the origin at center of mass of the body and inertia tensor I prime kl defined with respect to an origin in which with respect to an origin and which is not sounding good, origin or let me write it, it is not nice sentence but let me write it anyway in which I mean, I, what I mean to say is that you have chosen a system in which your origin is not at the center of mass, in which the, from which the center of mass is located at a.

If you are looking at the center of mass then from the origin it is at a vector is located at vector a. Inertia tensor defined with respect to origin of (cen) and defined with respect to an origin from which the center of mass is located at a are related as following. So, if you are looking at kl, I prime kl, then Ikl is with respect to the center of mass and then there is an additional term which is this where mu is the total mass of the body and then you have to have other terms. You see their two indices here so, whatever I write down should also have two free indices and this is what you will get.

So, if you look at this, this is as if there is a particle of mass mu located at this place vector a. so, let us proof, let us prove it. So, I will, just a second, so I start with Ikl which is defined in the center of mass, so you start with that one. Now, let us say this is my rigid body somewhere here is a center of mass, we are looking at some point p and this is my origin o and I am saying this is located at this place where this point is a vector a away from the origin.

So, with respect to the center of mass, so this point is center of mass, with respect to the center of mass, this location of the point p is given by vector r but the same thing from this origin is r prime, r prime. So, here is the inertia tensor in center of mass using origin as a center of mass. Now, what you do is, you substitute this that your r prime, r is r prime minus a, so your r is r prime minus a, let me put it in components so that is what you should substitute in this equation and when you do so you will generate lots of terms and some of them will look like what you should have in Ik prime and there will be other terms which you can combine with whatever you get in using these identities, these definitions.

So, m rk prime is the location of the center of mass times the total mass so you will get, you see, this is what this says this the entire thing divided by the total mass is the center of mass from this origin and where is it located? It is located at vector a. So, you should get this total mass times the location of your center of mass. So, this is all what you need to use and you will arrive at to get this, to get your theorem which is given in equation 1. So, please do that exercise and prove for yourself this relation. Now, let me add one more page.

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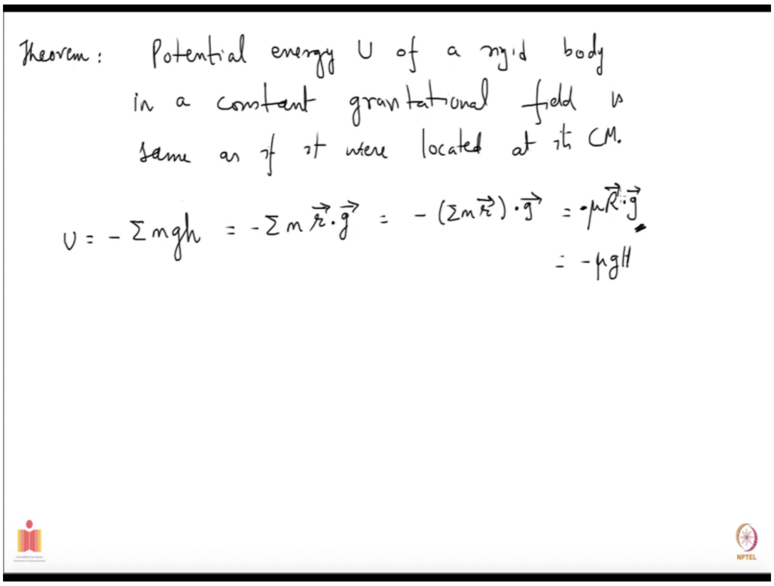

Now, I want to prove another which is trivial it says, that the potential energy u of a rigid body in a constant gravitational field is same as if it were located at its center of mass that is the theorem I am talking about. So, potential energy u of a rigid body in a constant gravitational field, field is same as if it were located at its center mass as if all the mass was concentrated there.

Let us see, how so you have the potential energy due to gravitation as minus I have to sum over all the particles ngh, the height of the those particles from some reference origin which is just minus (sum) summation over all the particles and let us say r is the vector which tells where the particle is and all you have to do is take a dot product in the direction of g so, you are looking at the component along the direction of g which will give you the h times g so, these two are same.

Now, g is a constant, so I can write down this as this I can take out and this you know very well what it is this is just this gives you the location of the center of mass times the total mass minus mu, mu is the total mass R dot gr is the radius vector of the center of mass and if you wish you can write it as minus, maybe I should write down in the next line minus mu gH, H is the height of the particle of the center of mass.

So, this is the next theorem that we will utilize, so from here you see that this is proving that as if your entire mass is located at center of mass the potential energy is the same as that one. So, that is nice. Now, let us go to our goal here which is to study the motion of a top in a heavy top which has which is under gravity, uniform gravity.

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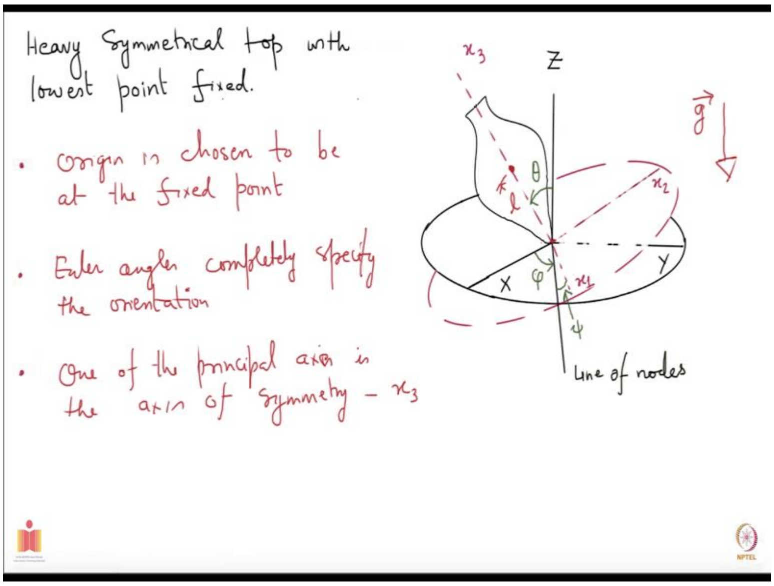

So here, I have drawn such a system let me write it down here. So, we want to look at heavy symmetrical top with lowest point fixed. We have all the ingredients ready for us we have already done everything that we need to analyze this motion. So, first thing is let me take some color or maybe here. So, few points the fixed point we choose the origin or just write origin is chosen at the fixed point. So, I am not going to choose origin at the center of mass which is let us say located somewhere here that is not where I am going to choose the center origin of this body.

I am going to choose the origin here. Second, because one point is fixed any orientation is going to be completely described by Euler angle. So, Euler angles completely specify the orientation. So, it is a simpler case in that context, then if you have to choose the principal axis of this body, it is easy to see what they would be because of the symmetry. One of the axis has to be along the axis of symmetry. So, one of the principal axis will lie along this it is going through the center of mass along this direction and that is what we choose to be x3 so that is the one of the direction of body coordinates.

So, let me write it down, one of the principal axis I think x is the axis of symmetry and that is what we are going to choose as x3 and you already know that any other I mean the other two will be in the plane perpendicular to x3 and that is what we are showing here by x1 and x2 so these are your principal axis which will diagonalize your inertia tensor. So, good so your x2 instead of the body axis now gravity is of course downwards here and center mass this place is located from here to there it is l, the distance from the origin to this place is l starting from here. I do not want to clutter it that is also good.

So, form here you will have a force mu g that is good. Now, one thing is that the origin is not at the center of mass so my inertia tensor would be different from the inertia tensor which we denoted by I, it will be I prime and I would like to know what I prime I have that is one thing. So, let me do that thing first.

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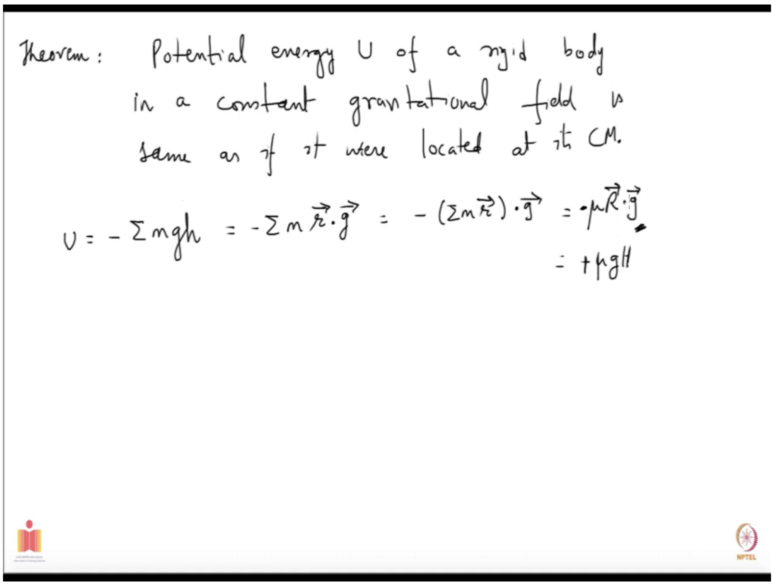

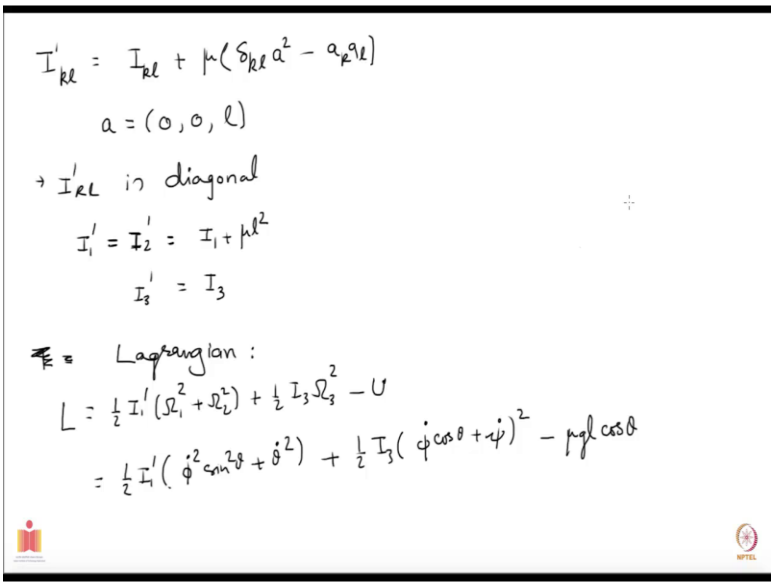

So, as we have seen, our I prime kl will be given by Ikl plus mu square no is it mu not mu square, it is mu not mu square. Did I write mu square earlier? No. It is correct. So, mu times delta kl a square minus ak al and what is vector a for me? Vector a is this. Can you guess what it is? It is along the x3 axis so it is component along x1 is 0 along x2 is 0 and along x3 it is l so, your vector a is 0 0 l.

Now, let us see 0 0 l if k is not equal to l if you look here in this expression if k is not equal to l, then the first term vanishes because the delta function vanishes and the second term vanishes because it is a product of two numbers, one of which is always 0 because it is either product of first one and third one or second one and third one and so forth and this will be 0. So, it is clear that I prime kl is diagonal. So, if you move the origin along one of the principal axis, if you keep it on there, it will still remain diagonal that is one point and then you would want to know what the diagonal entries are and I will give you as an exercise to check it is fairly trivial.

Show that I1 prime these are the principal, principal moments in principal moments of this body, I2 prime they are still the same, they do not become different and their value is I1 plus mu l square. Let us, check dimensionally, it is principal, moment of inertia has dimensions of mass time length square which is here also the case so it is good. And then check that I3 prime does not change it will still remains the same. That is also fairly easy to see. That is good.

Now, we would like to construct the Lagrangian for this system and I need to know the kinetic energy t for this and we have already seen what the expressions are let us go back here if you look at the kinetic energy due to rotation it is half I1 phi dot square sine square theta plus theta dot square and the second term. So, let me write down that down and I get t instead of t let me just write l no so my Lagrangian is a following, for the kinetic energy I have half I1 prime omega 1 square plus omega 2 square because I1 prime is same as I2 prime so I can do this plus half I3 omega 3 square minus the potential energy.

Now, this you can write as the following half I1 prime phi dot square sine square theta plus theta dot square plus half I3 phi dot cos of theta plus psi dot square and then you have minus mu gl cos of theta. I think this should be plus if your height is increasing that the potential energy should increase not decrease so this there should be a plus sign here which you can check by drawing a diagram you will see or from here also in from this diagram you can see so, this is what you get for the Lagrangian.

So, now we should analyze this system by looking at which coordinates are cyclic what are the first integrals and try to see if how much we can tell about this top and its motion based on just by looking at its first integrals. That is task which we are going to take up in the next video.