Introduction to Classical Mechanics
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Lecture 47
Angular velocity using Euler angles

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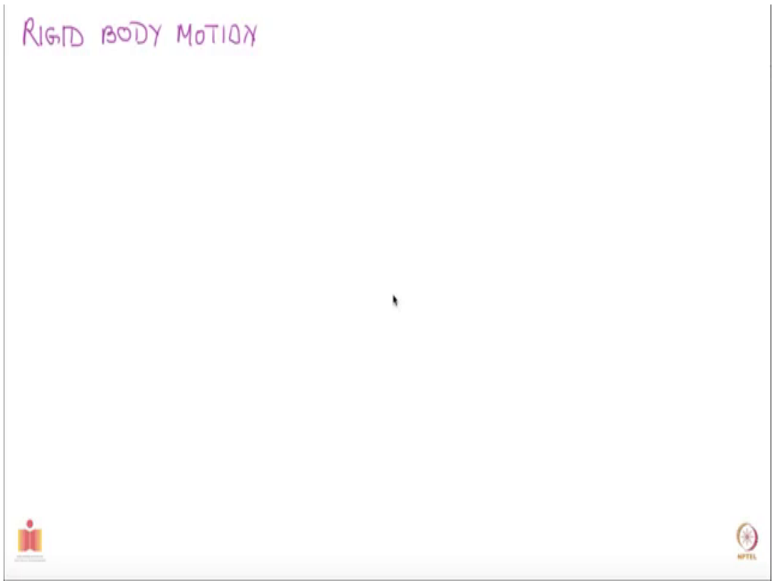

Let us continue our discussion of Rigid Body Motion. I want to start with writing down the angular velocity omega using Euler angles. Let us go back a few slides and visit the page where we wrote down Euler angles for the first time and try to recall.

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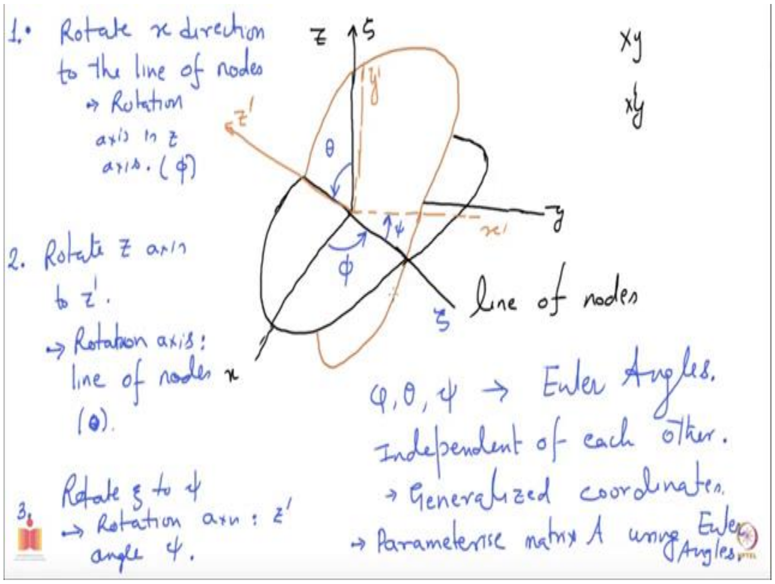

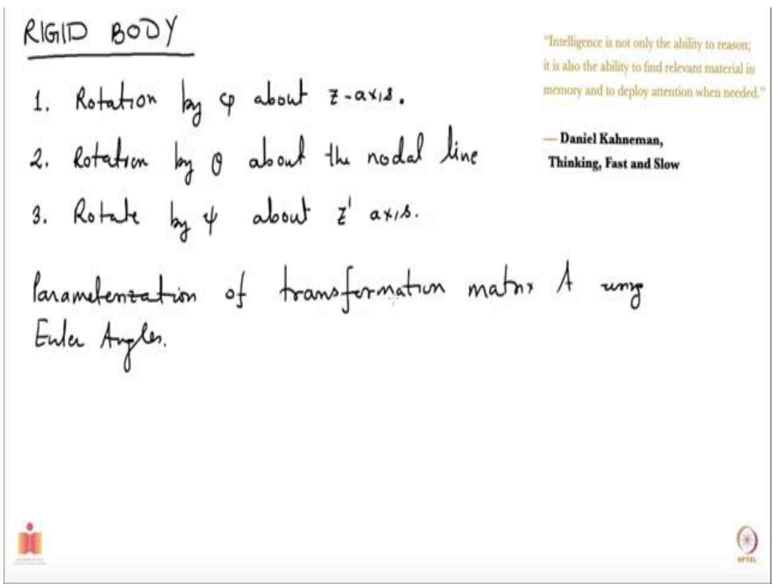

Somewhere here it should be. So, I hope you recall this, I have summarized here, let me write, see this thing. So, rotation by phi about z axis, so you see here the first rotation was about the z axis which was by an angle phi and then you have rotation by theta about the nodal line, so you took the z axis and brought to its new position z prime, by rotating by an angle theta about this line of nodes and then the final rotation was by angle psy about the z axis.

So, recall by after you turn by angle theta your z prime was at the right location and all you had to do was bring x and y at the right place and for that you needed to just do rotation by angle psy about the z prime axis. These were the things which define the Euler angles, theta, phi and psi and today we want to express angular velocity omega using these angles.

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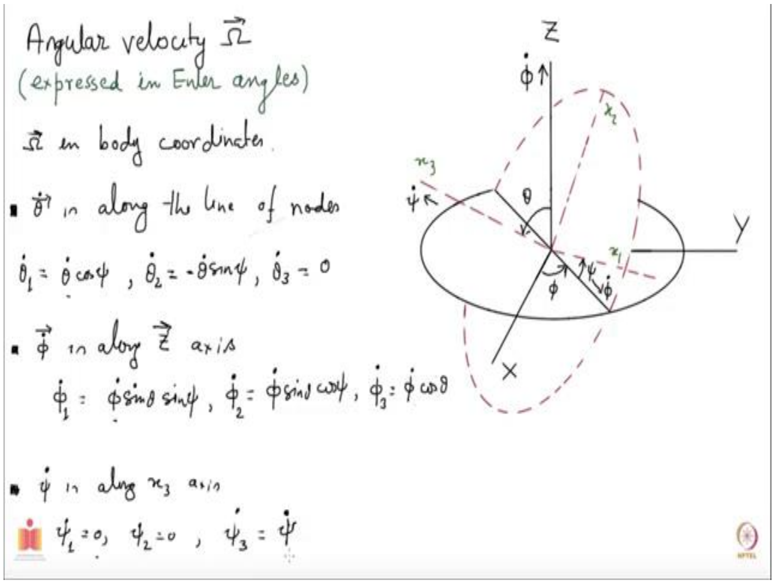

So, what I have done is I have redrawn before I started the lecture, the same figure slightly in more professional manner. So, your the black portion of this figure are your space coordinates, your X, Y and Z, capital X, capital Y and capital Z. So, let me write it down. No, I do not want to use pink, I want to use something else. So, these are your capital X, capital Y and capital Z, these are our space coordinates and then, let me write maybe, the dashed one, these are x1, x2 and x3. x3 is perpendicular to that plane made by x1 and x2 and these are the body axis, body coordinates.

Now we want to express angular velocity using Euler angles that is what we want to do. So, recall what I showed just now. Our first rotation was by angle phi about this axis. So, let me write it down here, by angle phi. So, when this, if you do a rotation about Z axis, then the angular velocity would be along the Z direction. So, your phi dot which is the angular velocity corresponding to this rotation will be pointing along the Z axis.

Then our second rotation was by angle theta and this was being rotated about the line of nodes which is here, this is the line of node. So, your angular velocity theta, theta dot is pointing along this direction, so let me put, theta dot here. So, theta dot points along this, your phi dot points along that and then your last one is by angle psy, too small, and this is rotating about the x3 axis, so your psi dot will point along x3.

So, these are the angular velocities which are pointing along these directions that is good. Now, if I want to express angular velocity omega in the body coordinates, I want to express omega in body coordinates, meaning I want to decompose omega along x1, x2, and x3. You could also depose omega along capital X, capital Y and capital Z, so that will be a decomposition in the space coordinates. So, let us start with first the body coordinates.

So, what I should do is, I should first find out the components of phi dot, psy dot and theta dot along body axis and from that I will extract omega 1, omega 2 and omega 3. So theta dot, let us start with theta dot, which is along the line of nodes as I said, along the line of nodes, meaning here. So your theta 1 dot, theta 2 dot, theta 3 dot, it will take too much space. So, let us write one by one, theta one dot. So, this is theta dot component 1, let us see theta dot where is x1, it is here.

So, I have to project theta dot along this, so it will just become theta dot cos psy. So, theta dot is the magnitude of this vector and you decompose it, so you get cos psi and for theta dot 2 you get minus theta dot sin psi and theta 3 dot is how much, see theta dot is along the line of node which means it is in the, in this plane and your x3 is perpendicular to that plane, so it is clearly 0. Next, let us look at phi dot. I do not know why I put vector here, it should have been on the top.

So, you can correct this one, you can write it as just like this. So phi dot is along the capital Z axis, remember because the rotation was about capital Z. So, good, let us look at phi dot. So, what you can do is first take phi dot and project it onto this plane, this plane which is denoted by the red curve, red dash lines. So, you can do that projection which will give you phi dot sin theta and the third component will be phi dot cos theta.

So, once you have projected on this plane you further project on x1 and x2 which will involve sins and cosines of psi, so here is your answer. I hope it is clear, phi 1 dot is phi dot sin theta, this projects onto the plane, this new plane and then you further project it along the x1 axis, because you see your x1 is just moved by angle psy, from this one you will have a sin psy here.

Now I write phi dot second component, again you have to start with phi dot sin theta and then you project onto the x2, you get a cos of psi and then you have the third component, phi 3 dot which is just projection along this and it is trivially phi dot cos theta. Let us see all looks fine, everything is fine and then we have the third piece, where is it, psy dot is along x3 axis, so it has only 1 component.

So, psy dot first component is 0, psy dot second component is 0 and psy dot third component is how much, psy dot. The entire vector is pointing in the third direction x3 axis. Now, how do I get my omega 1, omega 2 and omega 3? I should just collect the components which are, let us say I want to know omega 1, so I should collect all the angular velocity components that are in the direction of x1.

So, if I am looking at omega 1, I should take this, this, this and this which is 0. So omega 1 would be phi dot sin theta sin psy plus theta dot cos psy and omega 2 dot will be phi dot sin theta cos psy minus theta dot sin psy and phi omega 3, capital omega 3 dot would be, capital omega 3 would be phi dot cos theta, this is 0 plus psi dot. So, let me write down that for you.

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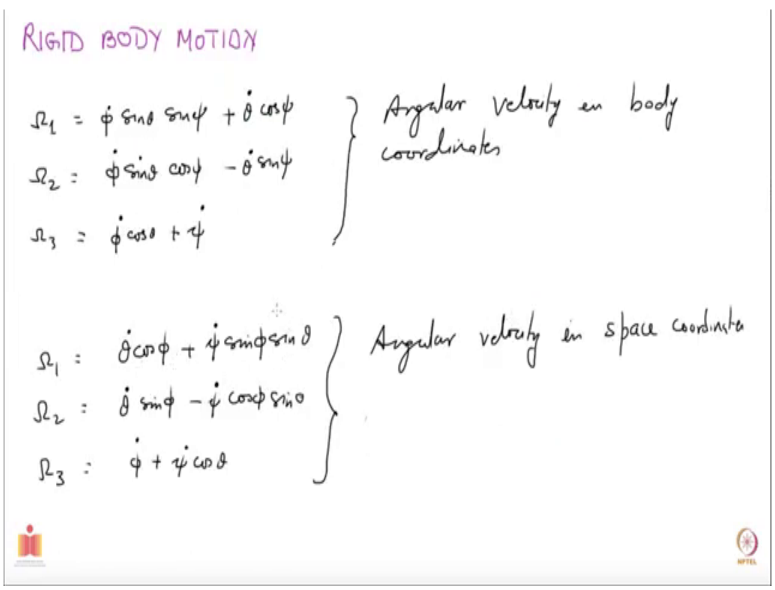

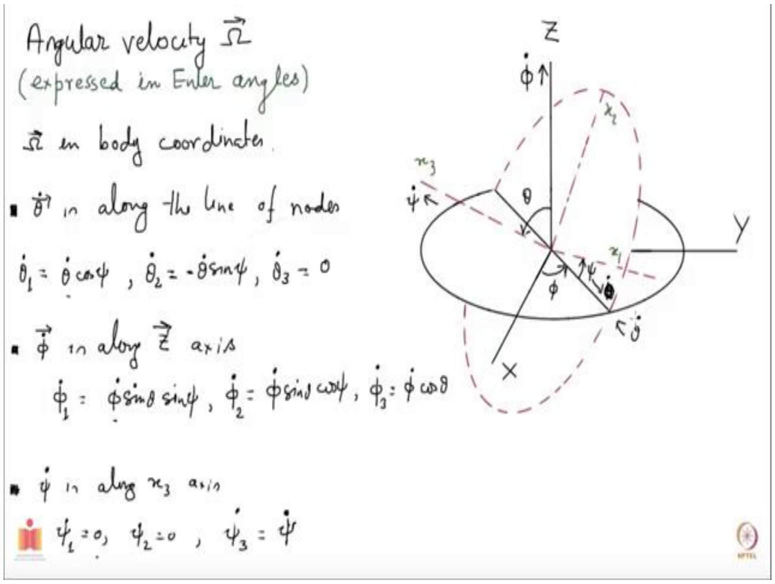

Omega 1 is phi dot sin theta sin psy plus theta dot cost of psy that is correct. Omega 2 is your phi dot sin theta cos of psy minus theta dot sin of psy and omega 3 is phi dot cos theta plus psy dot and this we have, what we have done is the angular velocity has been written in the body coordinates. But you could as well do this decomposition in the space frame, space coordinates in terms of X, Y and capital X, capital Y and Z and that will be also fairly simple.

So, I leave it as an exercise for you. Just repeat what I have done and decompose instead of this x1 you decompose along capital X, capital Y and capital Z. You should be able to obtain the following result. Let me write it down. You should get that omega 1 is theta dot cos of phi plus psy dot sin of phi sin of theta. Omega 2 should be theta dot sin phi minus psy dot cos phi sin theta and omega 3 should be phi dot plus psy dot cos theta.

This is your angular velocity in space coordinates or inertial coordinates, space coordinates. So, let us see everything is fine or not. For example, just I will show you one thing. For example, look at this one, theta dot cos phi, how does it arise, of course, because it is theta dot, it is corresponding to the theta dot and we are projecting it along the first axis, so it will arise from here. So, you have your, where is it, that is phi dot, something is wrong, this is also phi dot, this is also phi dot.

This should have been theta dot, this should be theta dot. Let me check. Sorry for the mistake. Let me try to use the eraser. I should not use eraser, I do not know how to use it. Let us go back to the pencil, everything is fine. So, anyway this should be theta dot, sorry for the mistake. So what I was saying was, because this is theta dot, instead of projecting it along those directions, if you project it along this capital X, you will get theta dot cos of phi.

So, theta dot will contribute this which is what you see here, theta dot cos of phi and similarly you can obtain all other terms. That is good. Now, the next thing what I want to do is, I want to write down the rotational kinetic energy.

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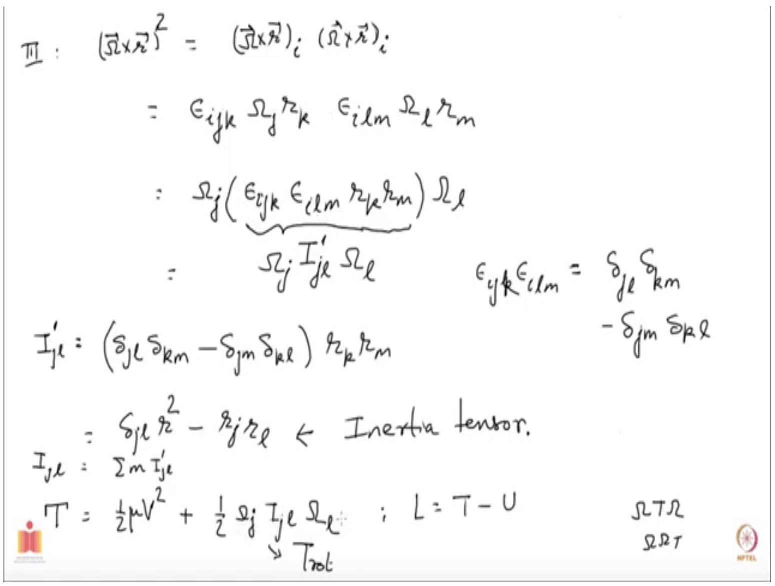

As you may recall we have written down an expression last time, somewhere here. So, this is a total kinetic energy of the system and this part I call T rotational, this piece, half omega i omega. By the way I hope you can, you have seen that this, if you are viewing it as vectors and matrices, this is a row vector, a matrix times a column vector, that is the multiplication, that is how multiplication, that is how this thing will appear in matrix multiplication, so if I write omega, T omega, it will not be like omega, omega T or something because you have to carefully look at the indices.

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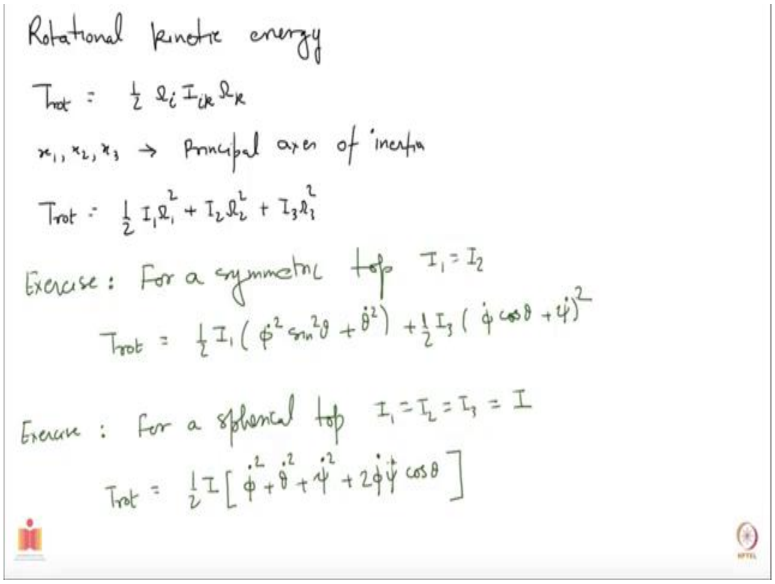

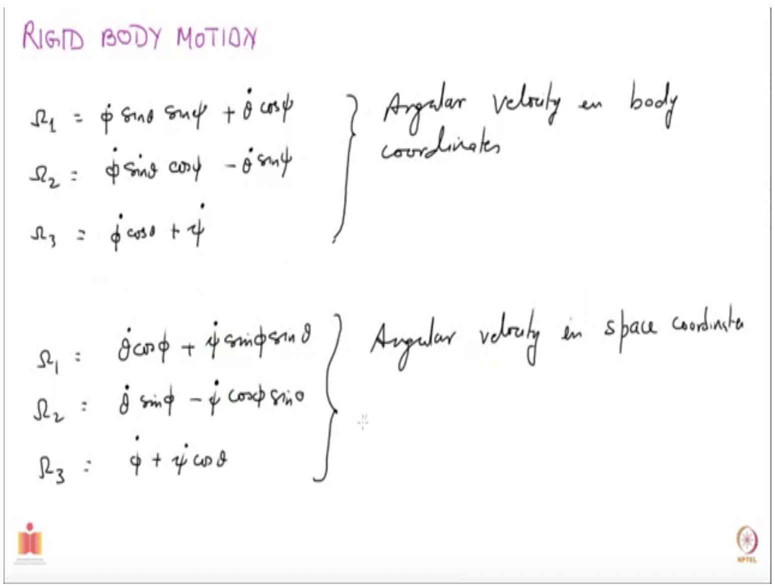

So, let us go to our rotational kinetic energy of a rigid body. So, as I said T rotation is half omega i I ik omega K. Now, what I will do is, I will choose the x1, x2 and x3 to be the principal axis of inertia. I can do so, I can always choose whatever I want. Now, once I do that my I inertia (())(19:27) becomes diagonal and in that case my rotational kinetic energy becomes I1 omega 1 square plus I2 omega 2 square plus I3 omega 3 square.

And we have already found omega 1, omega 2 and omega 3 in terms of our Euler angles here. So, we may want to write down the rotational kinetic energy for let us say symmetric top in terms of the Euler angles. So, I will give you a small exercise, it is fairly trivial. At some point I said that I will use green colour for exercises, let me try to do that. So, for a symmetric top show that, here I am using I1 is equal to I2.

Show, that the rotational kinetic energy is half I1 phi dot square, sin square theta plus theta dot square, these are coming from I1 and I2 terms and then the third term is I3 phi dot cos of theta that is correct plus psy dot square. Let us check, phi dot cos theta plus psy dot, phi dot cos theta plus psy dot, so this square gives this and the first two terms combine to give this one, I1 and I2 terms. All looks good.

And then you can check another thing, that if you are looking at a spherical symmetric top, spherical top where I1 is same as I2 and they all are equal to I then you get T rotational to be half I phi dot square plus theta dot square plus psy dot square plus 2 phi dot psy dot cos of theta. So you see that the kinetic term has cross terms also, phi dot psy dot is there. It is not diagonal in phi theta and psy. So, we will talk further about the rotational motion of rigid body in the next video.