Introduction to Classical Mechanics
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Lecture 46
Motion of a free symmetric top

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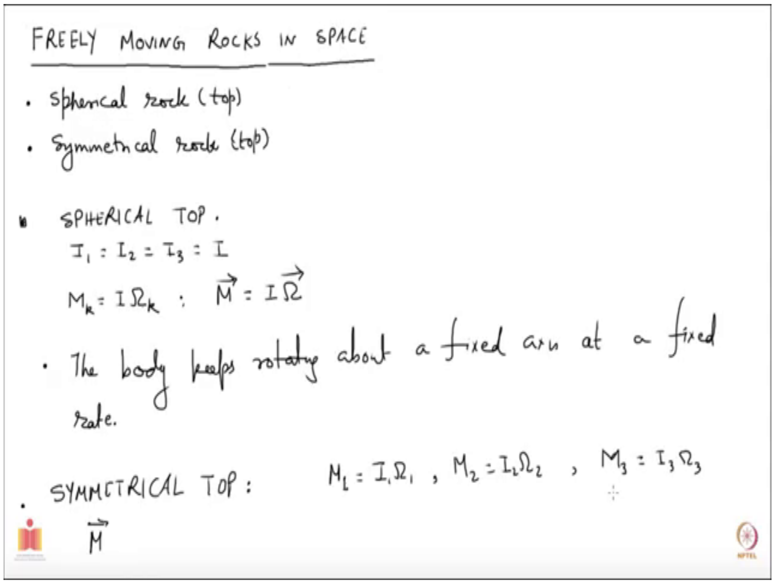

In this video we will start looking at the motion of rocks in space and these rocks will be considered as free that is there will be no force acting on them and as we have seen that these rigid bodies or rocks are generically referred as top and they come in different varieties, they could be spherical, completely spherical symmetric, so you say that is spherically symmetric rock as a spherical top.

If your rock is symmetric about some axis, you say you have a symmetric top and if it does not have any symmetry then you say asymmetric top. So let us imagine some rock is moving around in space and it is spherically symmetric and then we want to know how it will be moving about and as before, in the last video we would not be looking at the motion of its centre of mass. So, we are only looking at the non-trivial part, the part due to rotation of this rock.

So, first case of spherical rock is very easy to look at. So let me write down here, spherical rock. So, in this case as you know already all the principal moments of inertia are equal, let us say the common value is I and then if you want to know the intrinsic angular momentum M, let us look at the Kth momentum, K could be 1, 2 or 3, then this would be I omega K, that is what it is which means that intrinsic angular momentum is parallel to the angular velocity.

Because there are no forces on the top, your angular momentum is not going to change and which also implies that the intrinsic angular momentum is not going to change because the part which describes the motion of the centre of mass that will also be conserved. So, that would mean that the intrinsic angular momentum which we denote by M is also going to be constant in time, which means that the angular velocity omega is also going to be constant.

So, the direction, the axis about which the body which is spherically symmetric is going to rotate is not going to change with time. In general you know that omega vector is going to change with time, the axis of rotation is going to change with time, but for a spherical top because of its highly symmetric construction it is not going to change, which means the picture is this. The body keeps spinning or rotating about a fixed axis at a fixed rate.

So, that is the simple motion that it is going to execute. Let us now reduce the degree of symmetry that you have and let us say we do not have a spherically symmetric top but you have a symmetrical top, meaning your rock which you are looking at has only one axis of symmetry, so I want to look at symmetrical top and you can again imagine rock which is moving freely deep in space, it has been thrown in some manner and now you just want to know how it moves around.

So, let us see what we can say. If you look at the intrinsic angular momentum then M1, so I am imagining that I am decomposing the angular momentum of this body, whenever I say angular momentum I mean intrinsic angular momentum, so this M. Because there are no forces this is going to be conserved. So, let us say you think of a moment, one particular instant and at that instant you are going to decompose this M into its components about body axis.

So, let us say at a given moment the principal axis of the body are oriented in some manner and at that time you want to decompose M along those directions, so that is what we are doing. So, M1 will be I1 omega 1, M2 will be I2 omega 2 that is what we have seen already, it is the general thing and the third component will I3 omega 3.

Now, what I am going to do is look at where x3 points, so x3 is the, by x3 I am denoting the third coordinate which is attached with the body and I look at the vector M. So, vector M is going to be fixed in space and at that instant I am looking at x3 direction, so this direction of M and the direction of x3, they define a plane, any two lines define a plane or any two directions you can take and that will define a plane, so that is the thing which I am going to utilize now.

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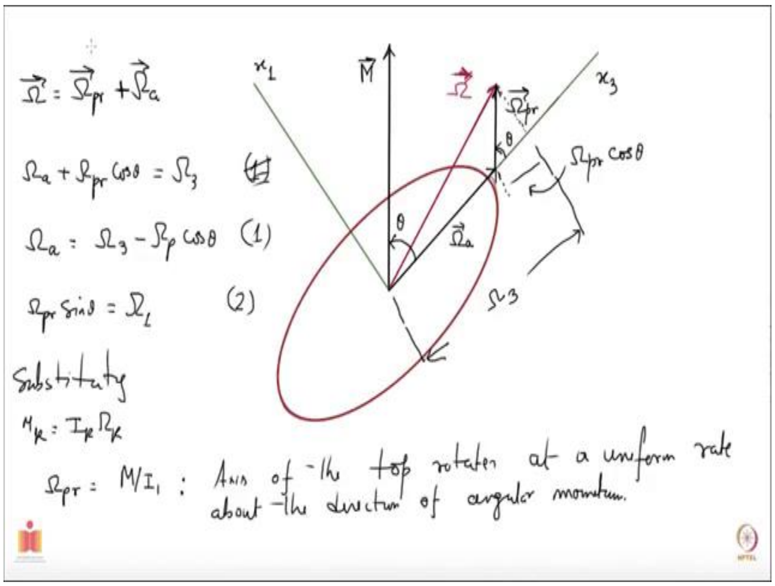

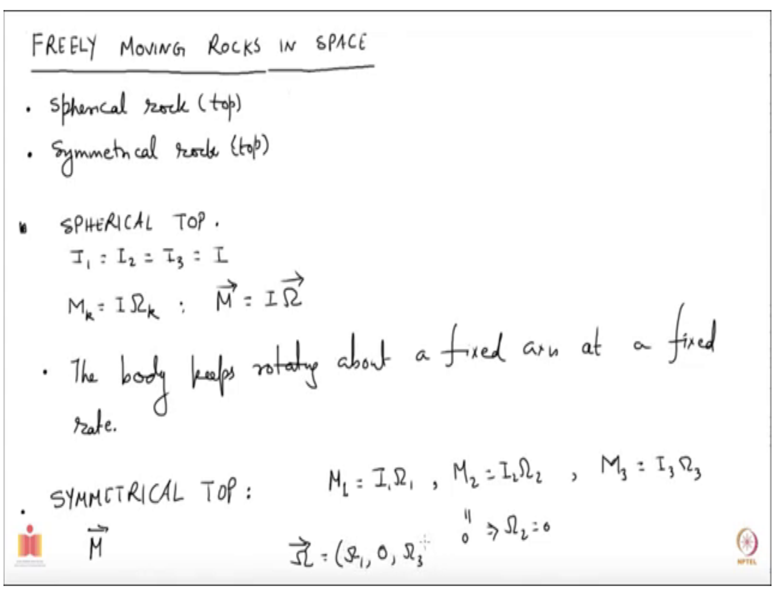

So, let us see here, I have drawn already this setup, do not look at x1, this I will explain. I wish I had not drawn this already but just to save a little bit of time I have done that. So here is the direction M which is not going to change with time, at the moment I am looking at things, let us say x3 axis is in this direction and I say that the plane which is defined by M and x3 is the plane of your screen, so that is the plane we have. So your M and x3 are in your screen.

Now how about x1 and x2, well they have to be perpendicular to x3, but because your body has symmetry about x3 you can take x1 and x3 to be in any directions, that is what we discussed earlier also. So what I am going to do is I will choose to put x1 direction in the plane of your screen and x2 will be of course perpendicular to the screen, so this choice I can make because of the symmetry or this object.

So, here we are, all the 3, x1, x3 and M they are all in the same plane at this instant. At the next instant of course things will change, but at the instant we are looking at everything is in same plane. So if I do so, then M2 is 0 because M is in this plane, M is in x1, x3 plane to it does not have a component M2, so there is no M2 component which means M2 is 0 at this instant, which means omega 2 is 0.

This is good which means, omega has only, omega 1 and omega 3 as the components and omega 2 is 0 which means your vector omega lies in x1, x3 plane. It is omega 1, 0, omega 3. This is a vector which is lying in x1 and x3 plane, so I want to draw that and of course as you know it will not point along M, so let me draw that, let us see if I can do it or should I, there is a, yes. Looks like it is going to work, perfect. I think this is good. Let me use some colour, maybe no colour, fine no colour. Let us use red.

So, this is the vector omega and omega is also going to be in this plane. That is nice, let me cross it. This is my vector and that is fine. Now, I want to first tell that the angle which axis of the top, this x3, x3 is the direction of the axis of the top, the symmetry axis. This makes an angle theta at this instant, let us say this is angle theta that is good. Now, let me draw a line parallel to M, so this should be vertical, let me maybe I should make a nice, now that works. So this direction which I am about to draw is parallel to intrinsic the angular momentum.

Let me, put just this looks perpendicular. So if I or maybe I just fill it up and good. So this one, I am going to call this vector from here to there as omega pr, pr for precision, you will see why I have used this subscript. So, anyhow it is just the projection of omega along the direction of M and then this remaining part, so this vector from here to there. Let me again bring this. This vector is what I am going to call as omega A. I call it omega A, because this is along the axis of symmetry. a means, subscript a axis of symmetry.

So, I5 have written my angular velocity vector as sum of these two vectors, where omega A points in the direction of the axis of symmetry and the other one points or is along the direction of M, so that is how I decompose it. That is good. Now let me see, so let us draw perpendicular here, so I project omega along this direction which will be omega 3, from here to there, it is omega 3.

Third component of the angular velocity and then this anyway is omega a and this piece from here to there is omega precision times cos theta because this angle is also theta. So this will be omega precision, omega pr cos theta. That is good now let us have understanding of this. So, firs thing is, let me write down a few questions now, I think I have written everything. So, first is that my omega is omega precision pr plus omega a, that is the decomposition of this vector we have done. So let me nothing, I think I have said already.

So, here omega has been written as a sum of omega pr which is parallel to M and omega a, which is along the symmetry axis of the top. So, from the figure it is very clear that you can write omega a plus omega pr cos theta, this is omega 3, so here you have omega 3 from here to there and this is sum of omega A and omega pr cos theta, that is what I have written, so let me call this as equation 1, very good and from this I can conclude that omega a is omega 3 minus omega P cos of theta.

Just taken one term on the other side and also your PR sin theta is omega 1. You see if you take this omega and project is along x1, this is what you get. It is equivalent to projecting your, it is equivalent to projecting your omega pr along x1 axis. So, your omega PR sin theta is equal to omega 1. That is correct. Now I know my M, angular momentum and I would like to determine these omegas and what I am interested is in omega a and omega pr.

So, I am going to use my omega 1 as M1 over I1 and omega 3 as M3 over I3; that is what I am going to use. So substituting that I get, let me write substituting M1, let us say Mk, Ik omega k, there is no summation over k. I get the following, I get you can easily check. That is trivial, that omega precision is M over I1. That is one result I have. So, this says that your top is rotating at a uniform rate about the direction of the angular momentum.

See the M is not going to change with time, it is going to always point in the same direction and you are looking at a vector omega pr which points along M which means that when you are looking at omega pr, it corresponds to the motion of the rigid body as a rotation about this axis, omega pr as the axis, so that is why it corresponds to rotation about the direction of angular momentum and that is what we call as precision.

So, let me write it down, this corresponds to the axis of the top at a uniform rate about the direction of angular momentum, about the direction of angular momentum and then the remaining thing let me talk about.

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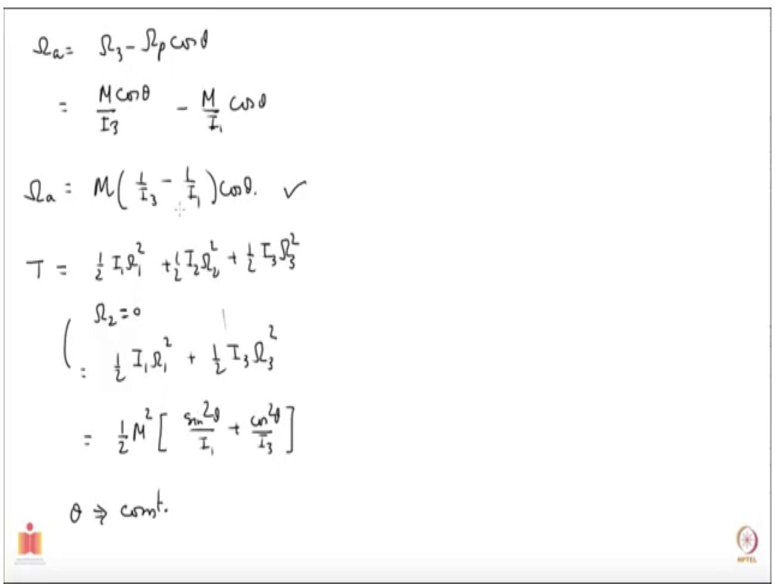

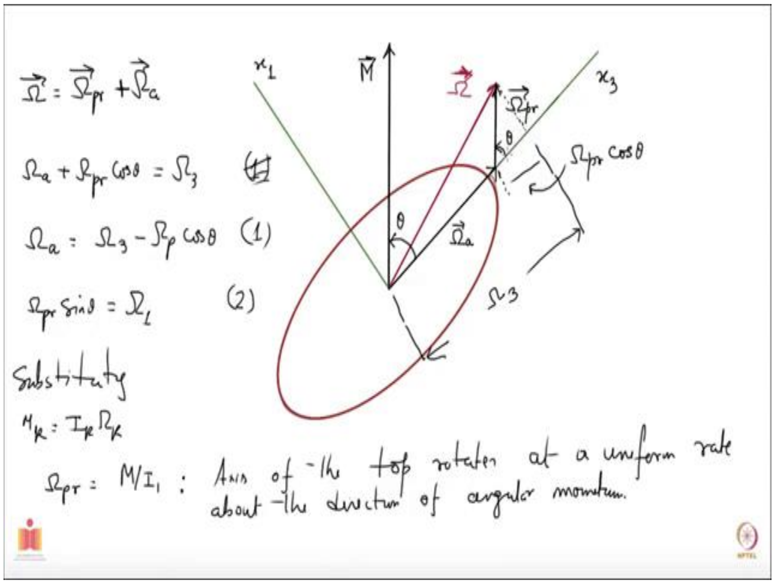

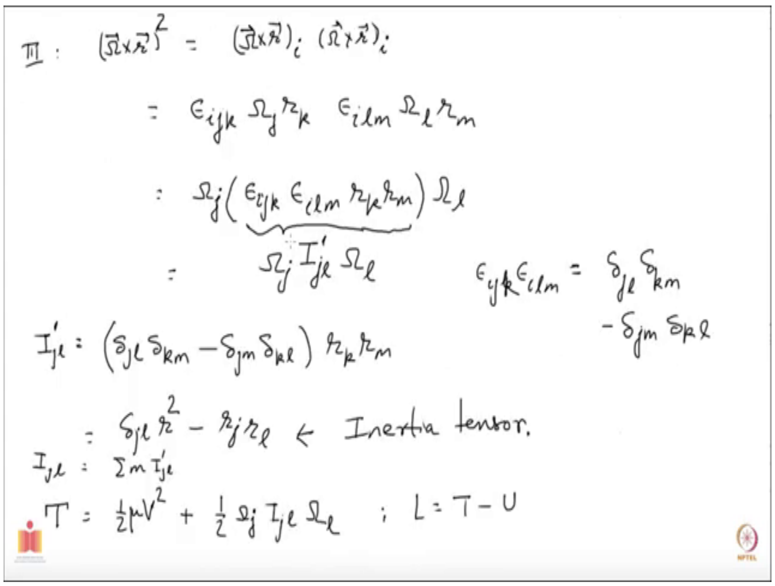

So my omega a is, where is it? I think I wrote it before is omega 3 minus omega p cos theta, so this is the equation I am using, so let me write it again. Omega Pp cost of theta, and what is omega 3. Omega 3 is your M3 over I3, but your M3 is M cos theta minus, omega p we have already found M over I1 and the cos theta factor is here anyway.

So, your omega a is M1 over I3 minus 1 over I1 cos of theta. So, that is the omega a which is in the direction of x3, so it corresponds to the motion of your top about the x3 axis which is basically the angular velocity of the top spinning about its axis; that is the spin of the top about its axis that is what your omega a gives and omega pr is giving you the precision of x3 about M.

Now I want to establish that the cos theta is a constant in this problem. So, for that I want to utilize, I would like to utilize the fact that the kinetic energy T is a constant because there are no forces on this body and I write T as half 2 square plus half I3 omega 3 square. I believe I have done this already, let me check, we were looking at correct.

So here you see we wrote down the kinetic energy as sum of these two terms and you had omega j, I jL omega L, all you have to do is now go to the principal axis, so once you are in the principal axis your I will be diagonal and it will have components I1, I2, I3 and this will become I1 omega 1 square plus I2 omega 2 square plus I3 omega square and each of the terms will be multiplied with the half. That is what I am using now, that is what I am writing now.

And the instant at which I am looking at everything, omega 2 is 0 remember that. Your vector omega lies in the x1, x3 plane. So, omega 2 is 0 which means your T is half I1 omega 1 square plus half I3 omega 3 square and if I again substitute what omega 1 and omega 2 are in terms of angular momentum, you get half M square, sin square theta over I1 plus cos square theta over I2 or I3. T is constant, M is constant, you conclude that your theta is constant.

Which means that the body is going to make a fixed angle with respect to M, the x3 is going to make a fixed angle with respect to M. This theta is not going to change. The rock is going to spin about its axis with angular velocity omega A that we have determined already here and it is also the axis, the x3 axis is also going to precess about M and x3 is going to move at a uniform rate about M, it will keep going like this and the angle will not change.

That is the general motion of a free rigid body which is symmetric. But here you note that the direction of M and omega were not same. If they were same, if they were in the same direction then what would happen, I will leave for that you to figure out, should be easy. That is good and maybe one more thing.

I will also leave you to figure out what happens if your I3 is smaller than I1 or if your I3 is bigger than I1 because there will be a difference in sin in these two cases. So, that is another thing I would urge you to think about. So, there is some motion of a symmetric rigid body in free space and we will continue more on rigid bodies in the next video.