Introduction to Classical Mechanics
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Lecture No. 45
Lagrangian of a rigid body

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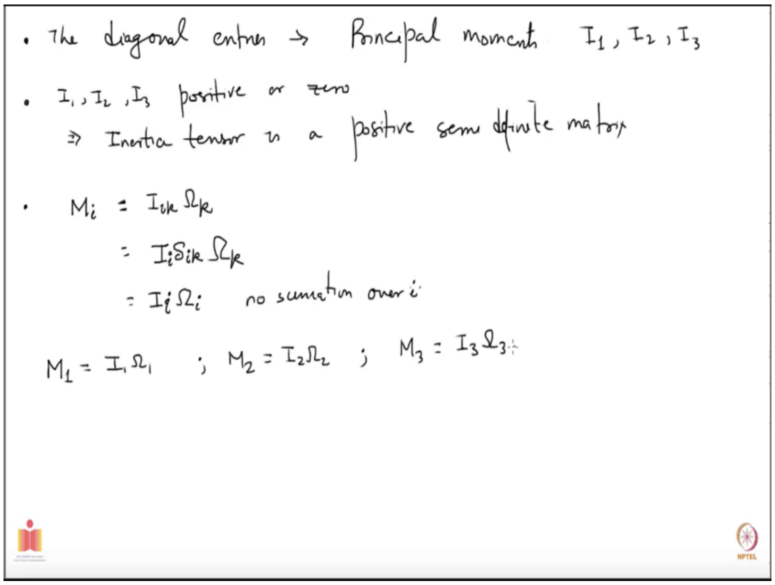

Let us continue our discussion of rigid body motion. Last time we introduced inertia tensor let me go back a few slides here and we wrote down an expression for the intrinsic angular momentum of the rigid body which was expressed in terms of the inertia tensor and the angular momentum or angular velocity omega k omega of that rigid body.

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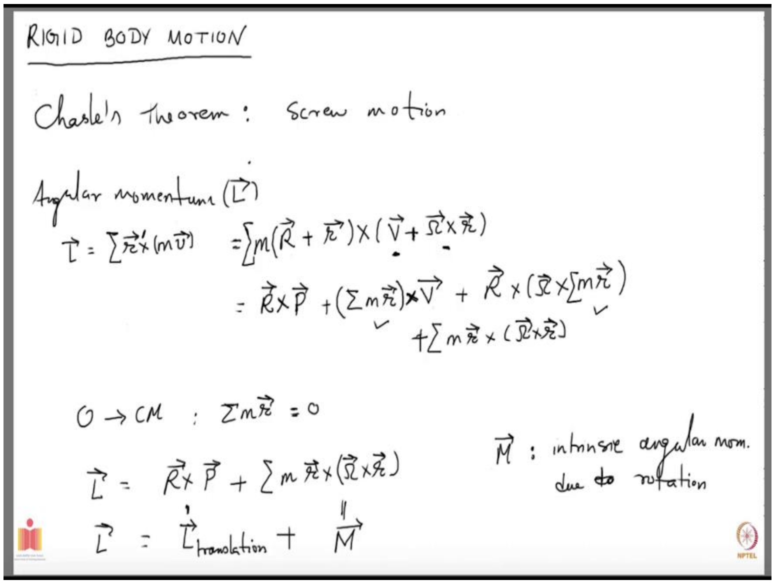

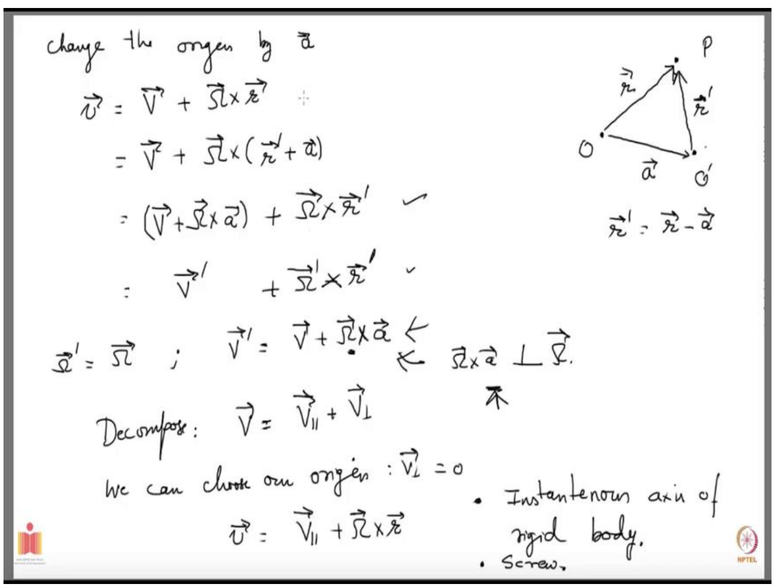

Now today we will start by writing what the kinetic energy is if I express this in terms of the angular velocity. That is the task we have in front of us. So, let me write down we are looking at the kinetic energy of a rigid body. Let me go back and see if I can here for example, if you see when we were looking at the angular momentum, the velocity V we wrote down as capital V plus omega cross r. here for example the first line. So, that is what I am going to use and V was the capital V was the velocity of center of mass. And we are going to take origin at the center of mass of the rigid body.

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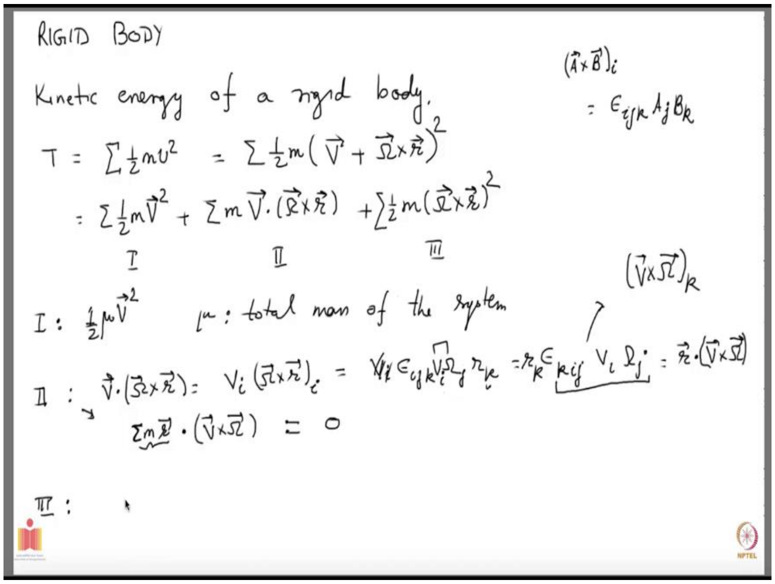

So, capital V would denote the velocity of the center of mass of the rigid body in the display system. So, let me write down the kinetic energy T is half mv square and I should sum over all the particles. If you wish you can put an alpha here and alpha here where alpha runs over all the particles in the body but because I am putting the symbols sigma make it implicit. I do not want to clutter. So, this is nice.

Now I can write down this as half m V the velocity of the center of mass plus omega cross r where this small r is measured with respect to the origin of the body axis that is what it is, square. If I open this up, I get half m V square plus m V dot omega cross r from here and then you get the third piece which is just the square of this. So, you get plus half m omega cross r. So, that is what we have omega cross r square. So I have got 3 terms 1, 2 and 3. Let us look at the first term.

You see the capital V can come out of the summation because that does not carry any label telling which point it is in the in the rigid body. So, this is a constant which will come out and it will become the summation over m would become the total mass so, the first term becomes half. Now the problem if I write capital M it looks like angular momentum half anyway let us put mu, mu v square, mu is the total mass not the reduced mass, that is fine. Then look at the second term.

Now here, you have capital V and capital omega which can come out of the summation because they are independent of which point you are talking about in the rigid body but r is the coordinate of the points and the summation runs over it. So, that when I cannot take out but the form in which it is written right now, I cannot pull it out. So, what I will do is, I will play around a bit with this dot and cross products and try to see if I can do something. So, that is the goal. So, I hope you already know this you might have learned in your Bachelor's that a cross product let us say you have A cross Be, and if you look at the ith component of this, this is written as given by epsilon ijk Aj Bk.

I hope I will have some time to talk about tensors and then I will talk more about these things but in case not then please study on your own. So, anyhow this is the, this is how you write down the components of a cross product and that is what I am going to utilize here. So, let us look at V cross v dot omega cross r. So, we have V dot omega cross r which means this is Vi then omega cross r i ok there is a summation over i. Now because this is what dot product is, then you have Vi, now ith component of omega across r would be epsilon ijk omega jrk. I, let me write down Vi here and remove this. So, that is the quantity we have.

Now what I want to do is, have r outside. So, what I will do is, create a cross product involving V and omega and then have r outside. So, what I will do is, I will ok fine. Let me just relabel the indices. So, i I will start calling as k and k I will start calling as i. So, this I will rewrite, you know this because these all these ijk are right now summed over right. There is a summation over them. So, they are all dummy and I can just rename them there is no problem. So I get epsilon kji, then you have that does not help at all. I was no, that is not something I should do. I will just do one thing. I will permute the index i and k.

I mean I will cyclically permute all these things so I will get epsilon kij and then you have let me write rk first this one, then you have Vi, then you have omega j which is same as r dot V cross omega. Is that correct? You see, this is what the definition of A cross B is. So in this case, the free index which means this gives you the V cross omega kth Index. So, this quantity here is V cross omega kth component. And then you have rk, rk times this thing, summed over k is the dot product, so r dot V cross omega.

So, I have freed up r from V and omega which means my second term becomes the following. So, I have summation mr dotted into V cross omega. And because the center of mass is that r equal to 0, this quantity is equal to 0 because this is what makes it 0, r is 0 for the center of mass. And your this is going to give you the center of mass. If you divide it by the total mass. That is good. Now let us look at the last term which involves omega cross r square. Let us go to the next page.

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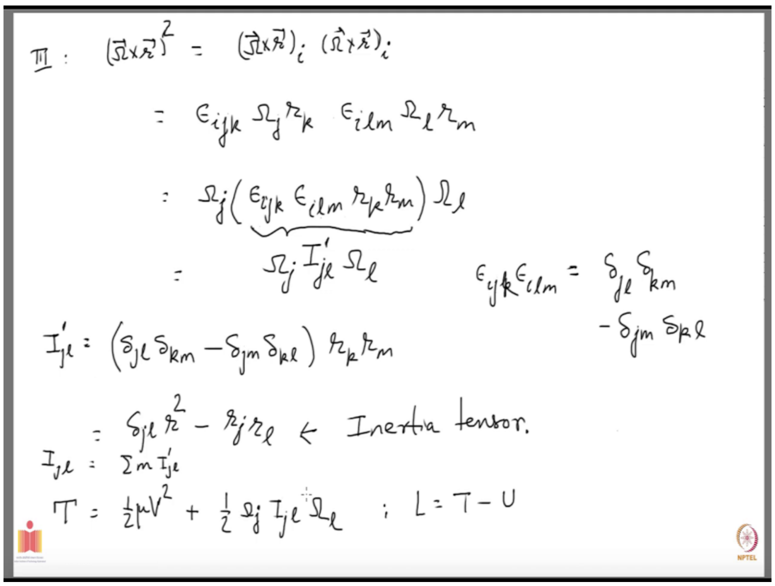

So third term, this has, this involves omega cross r square which is same as omega cross r ith component and again omega cross r ith component. And you have a summation over i and that is what the dot product is, but then I use this thing using epsilon tensor the live issue with the tensor you get omega epsilon ijk, omega jrk, then you have epsilon ilm, omega l and then rm that is nice. Now let me write it down like this omega j times epsilon ijk epsilon ilm, then you have rk, rm and then you have omega l. I hope all is good here.

Let us check ijk so, i is here repeat it so i is dummy, k is repeated k is dummy, m is repeated m is dummy. So, what is also j and l are repeated here but if you look at only this piece, let me call it some quantity i and what are the indices it is carrying? It is carrying j and then it has an l, others are contracted within this piece. So, with this I can write omega cross r squares as omega j, ijl omega l good. This ijl is again a second ranked tensor because it has two indices and these indices transform as vectors and that is why it is a second ranked tensor, this is nice.

Now I want to know more about what ijl is? Let us look at ijl as I have defined, let me use one identity. I am sure you are aware of it but let me write down ijl ilm so these are two (())(13:40) with our tensors and these indices first of first in index both are contracted. So, this is a quantity which has only 4 indices and not 6, so it is rank 4 tensor because there are only 4 free indices in here. And you know epsilon is fully anti symmetric under interchange of any two indices. So, if I interchange i with j it picks up a minus sign that is what one thing you know.

And I hope you are also aware that even if not, it does not matter to us but probably you are aware that epsilon is a invariant tensor. So, what we expect is that the answer will be a rank 4 tensor which will also be invariant and will also be anti-semitic under interchange of j and l and l and m, but any rate, at any rate here is the answer of this quantity. So, you have delta j with l, then delta l with m. This cannot be the answer because this is not anti-symmetric under jl interchange. What did I do? This is completely wrong. This should have been k that is why we had a problem ijk ilm. So, j with l that is fine that was good then k with m and to make things anti-symmetric under jk interchange, I should have j with m and k with l.

Let us check quickly whether it is anti-symmetric. If I interchange j k on the left hand side I will have a minus sign let us see whether I get here also. So, if I interchange j with k, you will get the first term will become kl jm which is what you have in the second term but there is a minus sign and you can check yourself that this term turns into the first term. So, there will be an overall minus sign which means the anti-symmetric property is getting reproduced. At any rate let us use it in above expression.

So, my ijl this new quantity is what is it, is delta jl, delta km minus delta jm, delta kl and then you have rk rm let us make the contractions. So, you have delta jl, then you have delta km when this is contracted with rk rm, it will become it will make it r square because this is this will become rk rk. And there is a summation over k so that is r square minus delta jm contracts with rm makes it rj, delta kl with rk makes it rl.

Can you see this is the same thing as you saw before, which means that your ij, i is moment of inertia tensor inertia tensor, not moment inertia tensor. That is our inertia tensor. Nice, so what is our kinetic energy now? The kinetic energy T has become half mu V square so that is the kinetic energy due to the motion of the center of mass so motion of the rigid body as a whole when it is translating plus there was another piece which has dropped out because of our choice of origin being at the center of mass.

If you do not do that, if you do not make that choice then that term will be present it will not be 0 which was here the second term. And the third term is half you can check. You have omega cross r square here. And then what we had originally was half m omega cross r. So I have to have half m in here. So, you have half omega, where is omega, omega j, then I should, maybe I should put the prime here. Because the ijl, the thing which has tensor it is inertia tensor has also summation over the masses.

So, now I bring in the this becomes ijl and omega l that is your kinetic energy and if you want to construct the Lagrangian, the lagrangian will be just T minus the potential energy here. And the potential energy will depend on your location of your center of mass and the orientation of your rigid body which will be given by the other angles which we have talked already in one of the previous videos. So, that is for the kinetic energy and we have again seen that inertia tensor appears here as well. And we can express nicely our kinetic energy as product of inertia tensor and the angular velocities. We will continue more on rigid bodies in the next video.