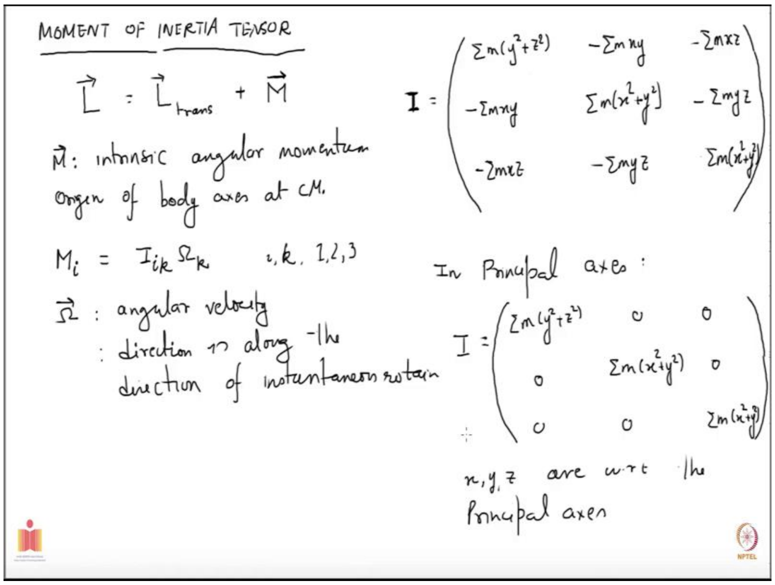
Introduction to Classical Mechanics
Professor. Dr. Anurag Tripathi
Indian Institute of Technology, Hyderabad
Lecture No. 44
Principal Moments

(Refer Slide Time: 0:14)



Last time we saw that we can split the angular momentum of rigid body into two parts, where one part is due to the translation of the center of mass. So, let me write it trans translation and other part is due to its rotation due to purely due to its rotation and that is what we call intrinsic angular momentum this capital M and this is what is our angular momentum. Let me write M is the intrinsic angular momentum. And the split we got because we chose the origin of the body axis to be at the center of mass.

So, origin of body axis, origin of body axis at the center of mass of the body and then also we saw that I can write down the intrinsic angular momentum M to be the following. It had the form Ii maybe I should write in components. So, if I am looking at the ith component of the angular momentum, then it is equal to this inertia tensor whose components are given by Iik and then you have the angular velocity vector omega k. There is a summation over k and i and k all run from 1, 2, 3, they take this value 1, 2, 3 and let me remind you our omega is the angular velocity of the body, angular velocity of the rigid body.

And what is the direction of this omega? This direction is along the axis of instantaneous rotation, is along the axis, along the direction not axis, along the direction of omega is along the direction of instantaneous rotation. You remember we proved Chasle's' theorem where we showed that the most general motion of a rigid body is that of a rotation about an axis which we call instantaneous axis and also a translation along the same axis. In some cases you may your body may not be translating and just purely rotating. But anyway so that is how you got the definition of instantaneous axis of rotation.

And then we wrote down the components of inertia tensor maybe I will spend 10 seconds writing that down. So, this is by I am denoting inertia tensor. If you do not like this you can put a bold face and let us recall what was there. We had summation over m y square plus z square. That was your first component of this matrix 3 across 3 matrix and if your body is not made up of discrete particles which will be mostly the case in which you will be interested, then of course the summation over m is to be taken as an integral over all the mass points.

So, you imagine a uniform density, let us say for your body mass density and then you integral over integrate over all the overall the body. So, that is what it will be and then you have minus summation m xy minus summation m xz minus m xy you remember it is symmetric so these two should be equal, mx square plus y square. Note that the diagonal entries which I am writing here they are always sum of squares. So, you have y square plus z square plus and this one is x square plus y square which means that your diagonal entries can only be positive or maybe 0 depending on how your body is made m xz, m yz and m x square plus y square.

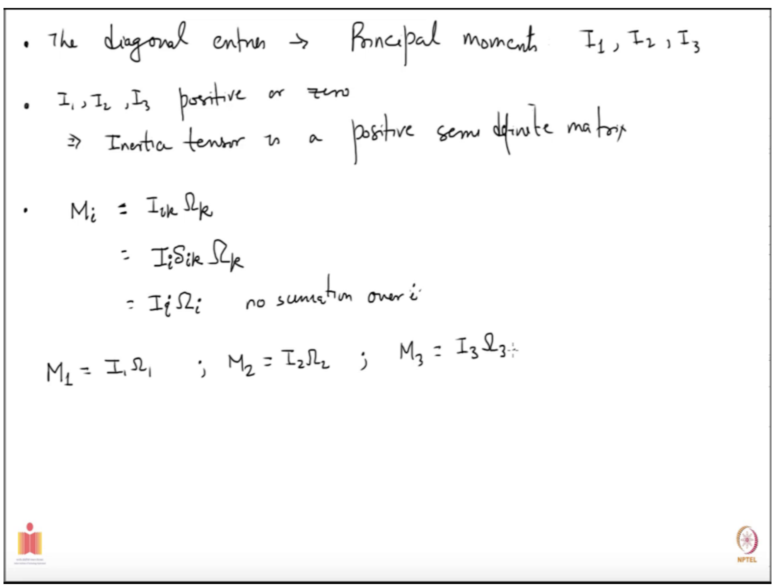
Your off diagonal entries are not enjoying their property that they are sum of squares, but your diagonal entries are anyhow. So, let us look at some properties of inertia tensor and as I said for those of you who have not ever studied tensors this is not going to prove to be of any handicap. I am not going to utilize any of the properties of a tensor in this. So, you for you it is a matrix. So, it is a 3 cross 3 real symmetric matrix which you know then you can diagonalize, meaning you can, if you recall what we did in when we were studying oscillations we just do the same thing.

So, I can choose I can do a principal axis transformation, meaning I can rotate my coordinate axis such that in new axis, this tensor would have only diagonal entries. So, I will be able to diagonalize this. So, in principle axis, so let us say my I have done a rotation of my coordinate axis and I have chosen a new set and again I am labeling the space the points of the body by xyz. So, I am not going to create some x prime, y prime, z prime I still use xyz but they are different from xyz here because I have changed the coordinates.

So, in this new principle axis your inertia tensor will be diagonal as you as you know already and what would they be? They would be my square plus z square 0 0 0 0 0 0 and then final entry would be, this is what you are going to get. I hope there is no confusion that this entry or whatever you get here let us say you calculate this quantity and whatever you have here they will be different. These are completely different quantities anyhow. So, now you see your let me, let me just write it down to be sure that there is no confusion.

Your x y and z are with respect to the principal axis. These quantities, these entries on the diagonal they are called principal moments of inertia. These are called principal moments.

(Refer Slide Time: 9:02)



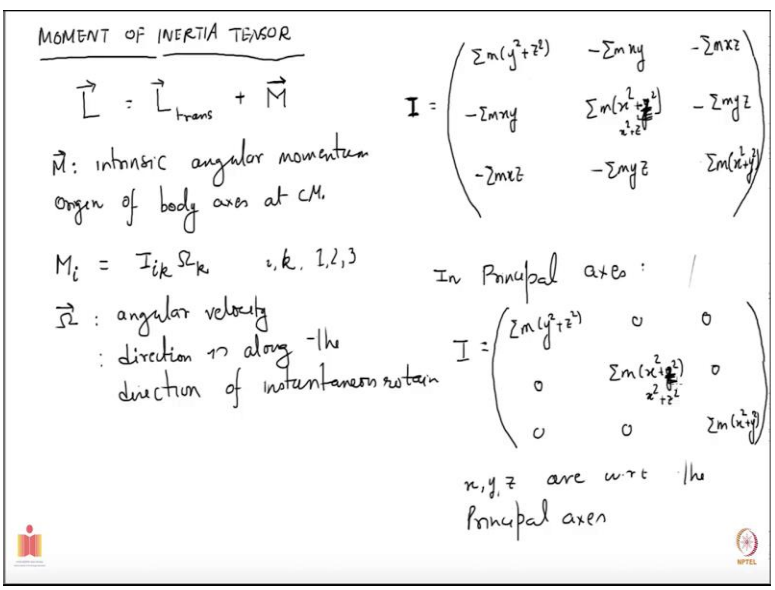
So, let me write down the diagonal entries. They are called principal moments. Generally they are labeled as i1, i2, i3. Then, as you have already noted, your principal moments can either be positive or 0. So, principle i1, i2, i3 positive or 0 which implies that your inertia tensor I is a positive semi definite matrix, positive semi definite matrix or positive semi-definite tensor. Now if you are working in principle axis, then let us ask how our intrinsic angular momentum looks like in this axis?

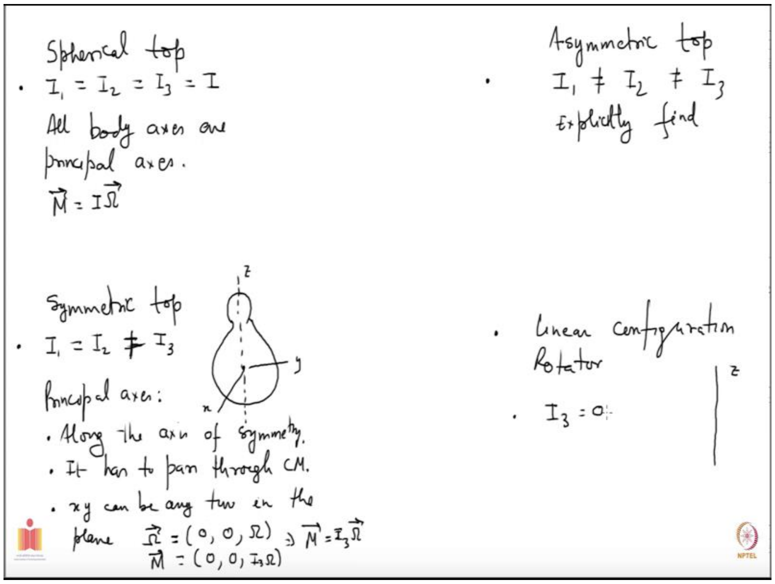
So, I am looking at that as component of angular momentum, sometimes I will not say intrinsic angular momentum I will just angular momentum because that is the only thing we are looking at right now. So, it will be Iik omega k that is the general expression but because we are using inertia, principle axis, my Iik is proportional not proportional but it has only diagonal entries so it will be delta Ik, this will be multiplied with Ii which are the principal moments.

So, if you look at this there is no summation over I here. Remember I is a free index on the left so it has to be free index on the right which means that I cannot have a summation over I even though it is repeated. So, it is not there. There is a summation over k which is implied. I hope this is clear because let us see this. Let us put I equal to 1. So, what do you have here? You have I1 delta 1 1, and this is what you should get here, I1 delta 1 k. So, you will get only diagonal entries for this matrix.

So, this is correct and with this I can write it as Ii and omega k times delta ik where k is sum I med becomes just omega i. And again I should emphasize there is no summation over i, no summation. So, which means that my angular momentum if I look at the first component it is just I1 times omega 1. If I look at second component, it is I2 times omega 2, m3 is I3 omega 3. Now clearly there are several possibilities for the values that I1, I2 and I3 take and let me list those down here. I should go to the next page.

(Refer Slide Time: 13:11)





So, there are several possibilities. Possibility number one is that I1 is equal to I2 is equal to I3 meaning, all the 3 principal moments are equal. Then the second possibility would be I1 is equal to I2, so two of them are same but not the third one sorry so third one is not the same as the first two but other two are equal. And then there is a, let me write down here possibility that your none of the principal moments are equal and there is a fourth possibility which is partly included in one of these is that one of the moments is 0.

So, let us call it i3 equal to 0. I will come back to this one so here I want to write one of the moments to be 0. Let us look at this first case when all the 3 moments are equal. You should convince yourself this is fairly easy to see that this corresponds to a spherically symmetric mass distribution. So, you have a spherical symmetry about the origin and such a spherically symmetric thing is called a spherical top. Now all the rigid bodies are tops everything is a is a top. So, generically they are called top so this is a case of spherical top.

This one is you can see that you should convince yourself that this corresponds to a mass distribution which is symmetric about one axis. So, for example you think of the top which with which you might have played in your childhood. So, that one has a symmetry axis which is if you are holding it in your hand, the nail of this top is giving you the direction of the symmetry axis. And you have symmetry around that and this corresponds to that configuration that mass distribution and it is not spherical but it has a symmetry so it is called symmetric top. And this one is asymmetric top.

Now how do you find the principal moments for a spherical top? Well, any any axis will be will be the principal axis because of the symmetry, if you choose a different set of axis, nothing changes everything looks the same. So, all coordinate axis all body axis are principal axis which means that independent of what body axis you have chosen you will always get the inertia tensor to be diagonal in this case. Let us go here. Now how do I find my principal axis? Well, it is clear, we should choose one of the one of the axis of the body to be along the direction of symmetry which in this case is along the z axis.

And the other two which will be perpendicular to this symmetry axis can be taken any any any two of them. So, any two which are perpendicular to each other will do the job because if instead of choosing let us say let us say this one corresponds to z axis so your let us see you have a symmetrical top which looks like, does not look very symmetric but you imagine that it is symmetric and it has a symmetry axis which is this the z-axis, it has to pass through the center of mass and you see there is a symmetry about this axis. Now you can choose any pair. If instead of this you rotate this entire x y axis to by some degree it will still serve as the principal axis because of the symmetry.

So, principle axis one of them should be along the axis of symmetry. Second, it has it has to pass through the center of mass. And then thirdly, the other two axis I will call them x and y can be any two in the plane. So, these two of course also have to be through the origin so look at this xy plane and if you do not want to use these two as your axis you can choose this and that any two. It will work that is good and here in this case of course there is no symmetry so you will have to explicitly find (plicit) explicitly find it out. So, it takes some coordinate axis, get the inertia tensor, diagonalize it and find the principal axis.

And let me so let us return back to the case of spherical top. For the case of spherical top, all these 3 are equal and I will call them to be I, do not confuse with your inertia tensor this is a scalar quantity just one number. And clearly my M is now I omega why because you see all these are same so it is just your angular momentum is proportional to omega. So, now in this case if your body is rotating about some axis the angular momentum will also be pointing along the same direction, along the direction of rotation. So, this is one special case.

Now here for this symmetric top you do not have you do not have this that your angular momentum points in the direction of, does not point along the direction of rotation or along the so for symmetric top in general, your angular momentum does not point along the direction of rotation axis that is not true, it does not happen but if you choose to have your omega pointing along one of the principal axis so let us say third axis.

You could have chosen first or second but let us say I choose angular momentum to be angular velocity to be this then your M1 M2 will be 0 because omega 1 and omega 2 are 0. So, it becomes 0 and only M3 survives and you get M3 to be non-zero which means your M as I said the first two components are 0 and only I3 omega will survive which means your M will be I3 omega in this case.

So, if you are going to rotate your body about one of the principal axis, then only the angular momentum will be pointing along that along that direction along the axis of symmetry, along the axis of rotation not otherwise. In general omega and m will point in different directions that is good. Let me also, one thing here I remember that during in the class students have difficulty imagining a top rotating in a direction different than the direction of symmetry axis. So, they were having difficulty in imagining that they can make the instantaneous axis of rotation for the top different from the symmetry axis.

So, they always have imagined or seen the top to be spinning about this nail, so they somehow always feel that omega can only point along the nail but I would encourage you to convince yourself that that is not true it is not the most general thing you can make your top rotate about an axis which makes some angle theta with the nail with the direction of the nail. This is one thing I would encourage and I would encourage that you do not try to see somewhere why this is true and spend some time thinking about it if this is not obviously clear to you.

And then we look at as I said the last thing as a object of a linear configuration. This is called a rotator. So, you think of a line which is let us say along z axis so you think a mass distribution which is kept along z axis, then let us go back and see. And of course you I am assuming that I have chosen the principal axis such that one of them is along z. In that case you will have z will be something but y will be 0 for all the particles in this line x will also be 0. Here is a mistake x square plus z square, did I make that mistake again? Here also.

Let us go back was it incorrect there that it was fine here. So this is this is x square plus z square this is also x square plus z square. So, because your body is only about z axis all the y and x components are 0 which means that this this piece I3 will be 0 and only I1 and I2 will survive. And this thing is called a rotator. So, now that we have some understanding of inertia tensor and principal axis, we will continue our discussion with I mean in the next time I would like to look at motion of a symmetric top, the top with which we have played in our childhood. It is simple motion but not in some gravitational field just in free space and that itself is having some interesting things to tell. So, see you in the next video then.