Introduction to Classical Mechanics

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Lecture - 43

Moment of Inertia Tensor

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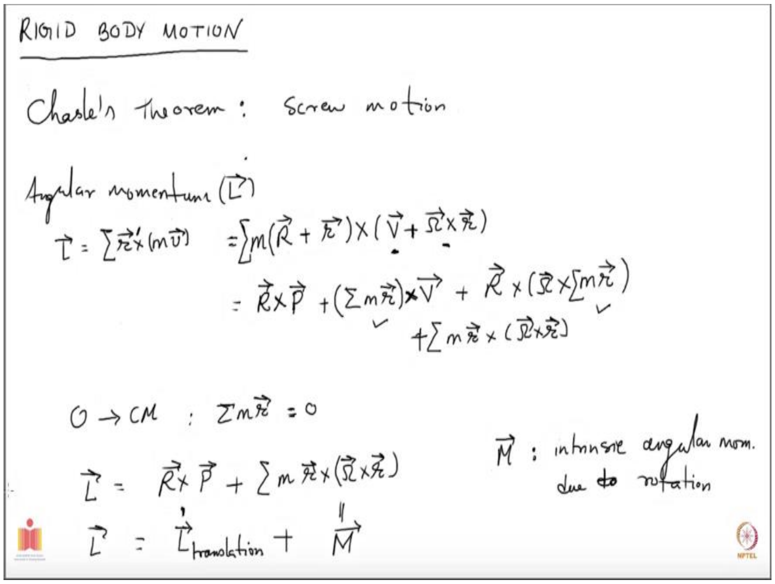

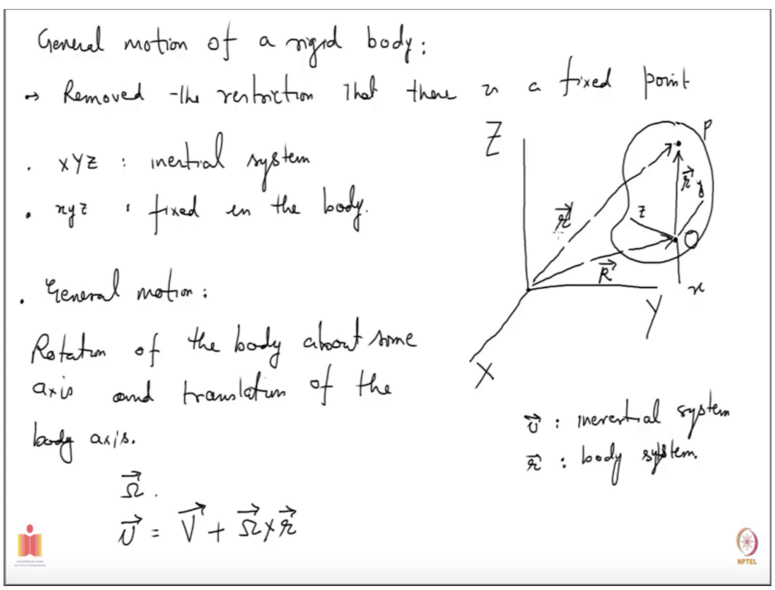

In the last video, we talked about the general motion of rigid body and we proved that the most general motion is given by the motion that is equivalent to that of a screw motion, meaning at a given instant, you can think of your body as rotating about an axis and also translating about that axis. And a special case would be that, there is no translation it is just rotating about that axis. And the theorem we did not name last time and I should say that it is called Chasle's theorem.

So that is what we saw. Let me write it down. Chasle's theorem, which the screw motion, that we saw last time. So today, we will start by looking at angular momentum of such a rigid body. So if I want to look at the angular momentum, let me, L, let me go back to this diagram here. So this is the origin. Let me denote this vector as R. The vector R gives you the position vector of the origin of the body system with respect to the origin of the space system or the inertial system. And this is the point P, which we are looking at and let us call this to be r prime.

So if I am looking at the angular momentum, then L is r cross p and p is m cross v, and I should sum over all the particles. I could use a label for, so I could put an i here, an i here, and i here and i here, and I will go I will run over all the particles. But since there is a summation symbol here, and I know that all these quantities are determined by this index i, I will just omit the index i. It keeps the notation uncluttered. So that is what it is. Sorry. It should be r prime. So I am looking at this the r prime is this one location of this one and in v is the velocity of this one.

Now, I can write this as, your r prime is R which is the position vector of the origin of the body system plus the vector r, which is relative to the body system. See, this r is with respect to the body system. There is the coordinate of the point p with respect to the origin here. So we will remember that r is with respect to the body system and this entire thing has to be crossed into mass times velocity. Maybe I will pull out the mass m here and velocity of a generic point is V plus omega cross r that is what we have already seen. Where V is the velocity of the origin of the body and here is our summation over all the points in the body.

Now, this will give you four terms because they are 2 times 2, four terms and I can write down. So first one is R cross V. Neither R, capital R nor this capital V depends on the label i. The only quantity in that product will be m, which will depend on i. So you get R cross V times the total mass, which makes the momentum of the center of mass, the momentum of this body as a whole. So that is the first term.

Then you get m, this small r cross V. So let me write down summation m r and this entire thing crossed with V. V does not depend on the, there is no i index on V or body index whatever, so then this sits out. This, it sits outside this. And then, you have two more terms. Then you have R cross omega cross r, r. I will bring out, bring the m together with that because only these are the two pieces which depend on where those particles are located because I plot the summation symbol there plus one more term, the most important one and that is m small r cross omega cross r. We have to sum over all body points.

Now, if I take the origin of the body system to be at the center of mass. Meaning I if I take O to be at the center of mass of the body, then because this vectors, small vector r is relative to that origin then summation over m r will give you the location of center of mass which will be 0, which will be 0.

So let me write down this, that is the effect of choosing origin of the body system at the center of mass. So this term goes away, that term goes away, and you are left with only two pieces. So let me write it down. You have with this choice, angular momentum as R cross P, which you recognize as the angular momentum of the center of mass because R is the location of the center of mass and P is the momentum of the center of mass with the total mass concentrated there plus you have piece m r cross omega cross r.

So this piece is really due to the rotation of the body, right? Because this is what this omega cross r gives you. This part, this part was for the translation. This part was for the pure rotation about the rotation axis. And so, this piece is the piece, which is purely due to rotation and we call it as intrinsic momentum, intrinsic angular momentum and I will denote it by M. And this is you can call it L prime or whatever, L translation. This is due to the translation of the center of mass. So my angular momentum L is sum of these two terms.

So let me write it again. M is intrinsic angular momentum and this is due to, purely due to rotation. No translation is involved in this, the translation part is here. Okay, that is good. Now, let us remember what it is m r cross Omega cross r.

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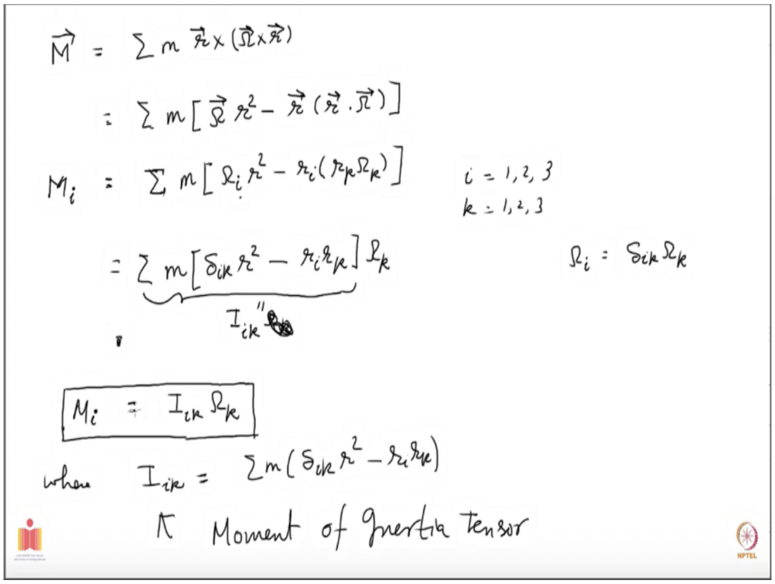

So let me write down again. My, sorry, intrinsic angular momentum of the rigid body M is given by summation m r cross omega cross r. And remember, all these vectors small r are relative to the origin, relative to the body system. So if you put r equal to 0 for example here, you are at the origin of the body system, not at the origin of the space system.

So nice. Now a simple identity you can use and convert this to the following. You can write it down as m omega r square, where r square is x square plus y square plus z square of that particular particle. And you have to sum over all the particles minus r and then you have r dot omega. Please check that this is correct, what I have written down. Now, I will do a little bit more and write it down as the following.

Now, I will take the ith component. Let us say first component or second component, ith component of the intrinsic angular momentum. And now, you can understand why I did not write the index on these, the particle labels, because I am going to introduce one more label and it becomes ugly if you put too many indices on this. But anyway, this sigma is going to remind us about the body index.

So m is there, no problem. This is the only vector quantity in here. So you pick up the ith component. Then you have the scalar r square minus r i. This is only vector quantity and r dot omega is a scalar, so there is no problem. I can write this as r k omega k, Einstein summation convention is in force here, so because the k it is summed over, it gives you the dot product. And i takes values 1, 2, and 3, and the k also takes values 1, 2, and 3. These are x, y, z components basically. That is good.

Now, I will write down omega i as delta i k omega k. I am doing so because then I can pull out omega k because this one has omega k. So this I get as this m delta i k r square minus ri rk times omega k. Now, see this omega k is in, I mean this is for the entire body. This is not specific to this point or that point in the body. So this does not care about the index which this summation, let us call it alpha, this carries. Those indices are only on the rs.

So I can write it down as, if I defined this thing as I, now how many indices this should carry? If it was a vector, it would have one index i but you see it is something else, it has two indices. So let me write down two indices here, i k both the terms have two indices that must have times omega k, where this is this. So I write this as m i is I, sorry. This piece is Iik omega k, where Iik where this capital I is defined to be m times delta ik r square minus ri rk. And this is called the moment of, is called moment of inertia tensor.

It is not a scalar. It is not a number. It is a tensorial quantity, it has two indices. If you have never seen a tensor before, I would encourage you to read some book. It is fairly trivial quantity, or maybe I will probably make a short, devote some minutes to this in a later video. But in case I do not do it, please read some book.

So in any case it is a tensorial quantity, but even if you do not know what a tensor is you can still see that this is a matrix. Okay a 3 cross 3 matrix. So it has an index i, index k they all run over from 1 to 3. So you have nine elements in total and this is a 3 cross 3 matrix. So what this matrix looks like, before that let me just emphasize this formula that the angular momentum, ith component of the angular momentum is given by this product of inertia tensor times omega k. That is good. Let us look at this property.

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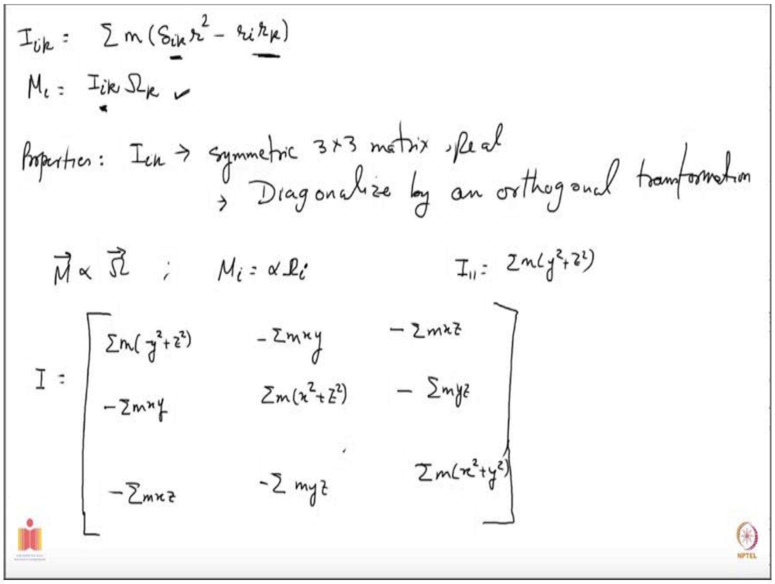

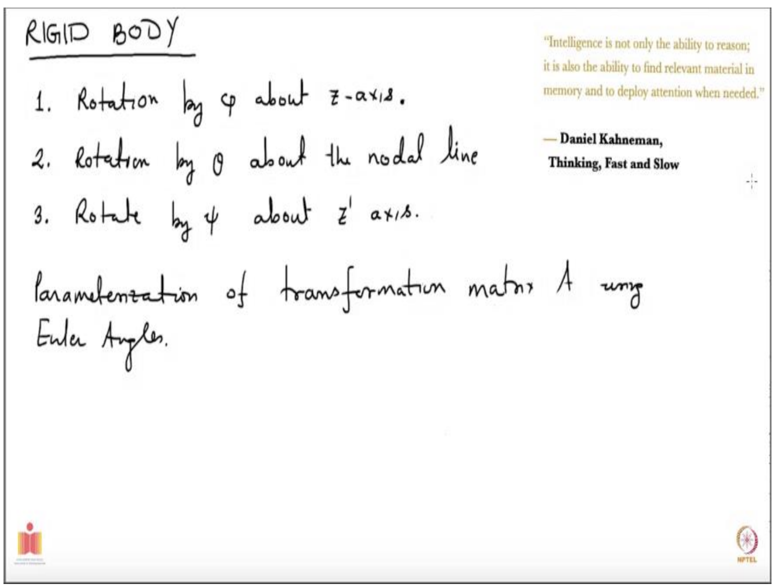

So my, let me write down here again, Iik, moment of inertia tensor is defined as summation over m delta ik r square minus ri rk. Is that correct? That is correct. And Mi as, perfect. Now, properties, clearly if you interchange k with i, you look at Iki, it is clearly equal to Iik because this is symmetric under the interchange of i k, this is also symmetric under the exchange of i and k, right because it becomes rk, ri, which is same as ri, rk.

So I, let me just, this is a symmetric tensor or symmetric matrix 3 cross 3 matrix. It is real, right? All the entries are real. There is no complex quantity entering here. Does that remind you something? If you have a symmetric matrix, real symmetric matrix, you can do something. You can diagonalize it. So to this, we will come back a little later and you can diagonalize it by an orthogonal transformation.

Now, let me go back and I wanted to show that quote again, which I wrote down. I found this very nice quote somewhere. Yeah, here.

Let me read out this. This is one of my very favourite books "Intelligence is not only the ability to reason, it is also the ability to find relevant material in memory and to deploy attention when needed." So the relevant material at this moment is that this is a symmetric matrix, and it is diagonalizable by an orthogonal transformation. And this is what we encountered when we were looking at oscillations. So I will come back to this again. Let us for the moment, see a little more about the inertia tensor.

So clearly, it is not true that the angular momentum or the intrinsic angular momentum of the body would be along omega, the direction of omega. Remember, what is omega? The direction of omega is the direction of your instantaneous axis. So if at a given instant, your body is rotating about some axis, this relation is telling you that in general, the angular momentum will not be along that direction, which is contrary to the experience most of us have had in our education if we have already not encountered this before.

So most of the time, you are, if you have not seen this before, you have only seen those cases in which the body has angular momentum in some direction and the object is also rotating about the same direction.

And now, I am saying that this is not the case because you have a much more complicated relation here. And if you are not very clear what, why I am saying this, let me tell you if your angular momentum and omega have to be in the same direction, then your m should be proportional to omega, right? It will be proportional to this vector, which would mean in terms of components that Mi would be some constant times omega i. This is what the relation would be. But what you are seeing is not this kind of relation but something different.

Fine. Now, you may ask why you have seen that before that your omega and M always going in the same direction. It is because you are looking at very special geometries of rigid bodies and those objects were also made to rotate along some very specific axis. And that is why you are always seeing M to be proportional to omega in your previous experiences.

Now, we will come back to all this later in more detail. But for now, I just want to write down the most general expression for the inertia tensor, I. So let us look at this. Let me look at I11. So what is I11, 1 1 so I put the i and k to be 1 and 1, which means this delta 1 1 is 1 r square, which is x square plus y square plus z square of that particle minus r1 r1, which is x x. So you have x square plus y square plus z square minus x square, which makes this y square plus z square.

So you get summation over m y square plus z square. So similarly, you can find all other components and let me write down there for you. You get summation over all the masses, all the particles and this, not just a masses but with this product. And then here, you get minus m x z, then you get minus m x y, m x square plus y square, sorry, z square minus m y z, x z, y z and then, you have the last entry to be m times x square plus y square. Okay, good. Looks symmetric. Let us see. Everything looks fine to me.

So we will stop this video here and we will talk more about moment of inertia tensor and rigid body motion in more detail in the next video. See you then.