**Introduction to Classical Mechanics**

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**Lecture - 41**
**Rigid Body, Euler's Theorem**

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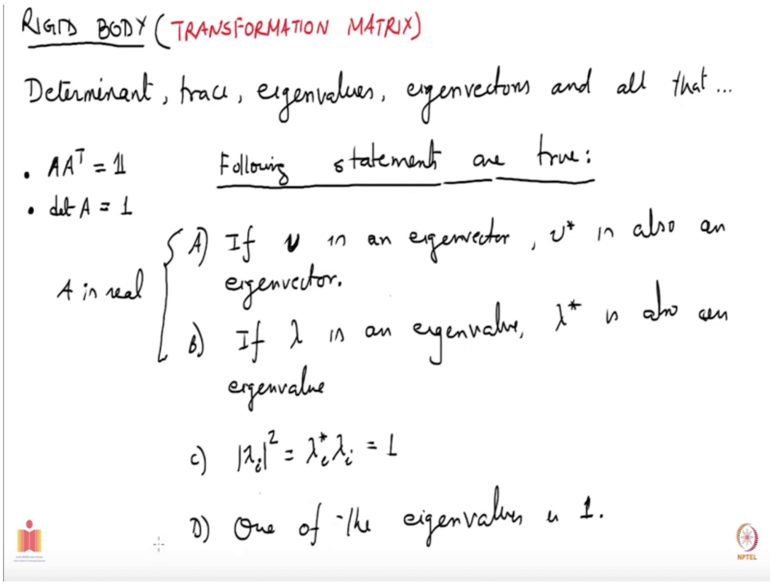


Let us continue our discussion on Rigid Bodies. Last time, we were looking at transformation matrix and as I have said on previous occasions, whenever we see a matrix, we should immediately ask about what its determinant is, what is its trace, what are the eigenvalues, eigenvectors, and such questions we should ask, okay. And that is what we are going to do in this video.

So, I will give a subtitle to this just to emphasize that such questions we should be asking. Determinant, trace, eigenvalues, eigenvectors, and all that. Let us see whether asking about such things gives us some more information. Now, first thing we know about our transformation matrix is that it is orthogonal. AA transpose is 1 or unit matrix, it is orthogonal.

Second thing, we have already found is that the determinant of A is unit, it is 1. And let us see what we can conclude from these two properties of these matrix about the eigenvalues and eigenvectors. That is what I want to do in this video. Also, remember that A is a real matrix, it has no complex entries.

Now I want to make the following claim. So, I write, following statements are true. So, first is that if V is an eigenvector, you can put it in boldface if you wish. I will not put, I will put it only once. If V is an eigenvector, meaning it is a column, which satisfies the eigenvalue equation. V is an eigenvector, then V star is also an eigenvector. Meaning that if you solve the eigenvalue problem, and you have found, let us say, one vector then you can close eyes and say if I take the complex conjugate of this vector that will also be an eigenvalue of matrix A. So that is claim number 1.

Claim number 2 is if lambda is an, is an eigenvalue, then lambda star is also an eigenvalue. In fact, what is true is that if lambda corresponds to V, then lambda star corresponds to V star. Also, we will show that all the eigenvalues, let us call them lambda i, where i will run from 1 to 3, they all are unimodular, meaning if you take a mod square, which is same as lambda i star lambda i then this is 1. Star is for complex conjugation.

And we will also conclude that one of the eigenvalues is unity. So, at least one of the eigenvalues is 1. So, let us try to see that if these are true. Let me also tell you that for these all I will need is that matrix A is real, and then maybe this is fine. But I will, it will be better if I tell you from what these things are following. We will put it later after we have derived it. Okay, so let us do it this way.

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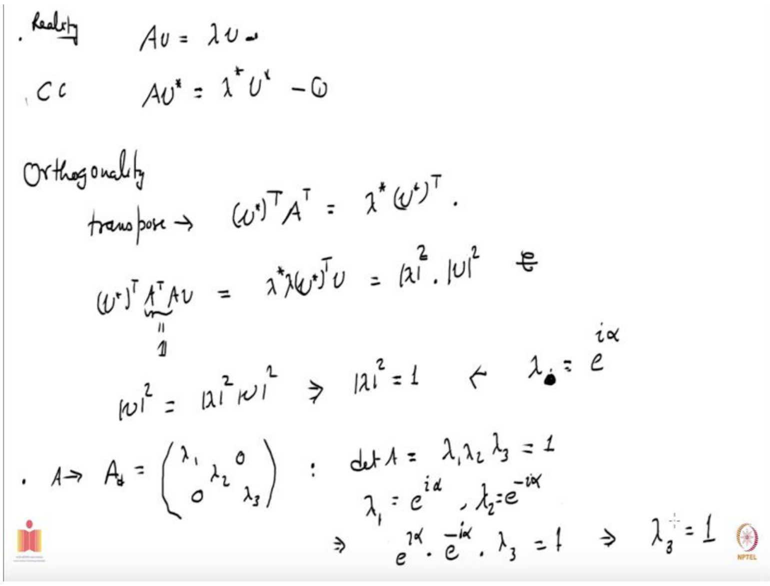

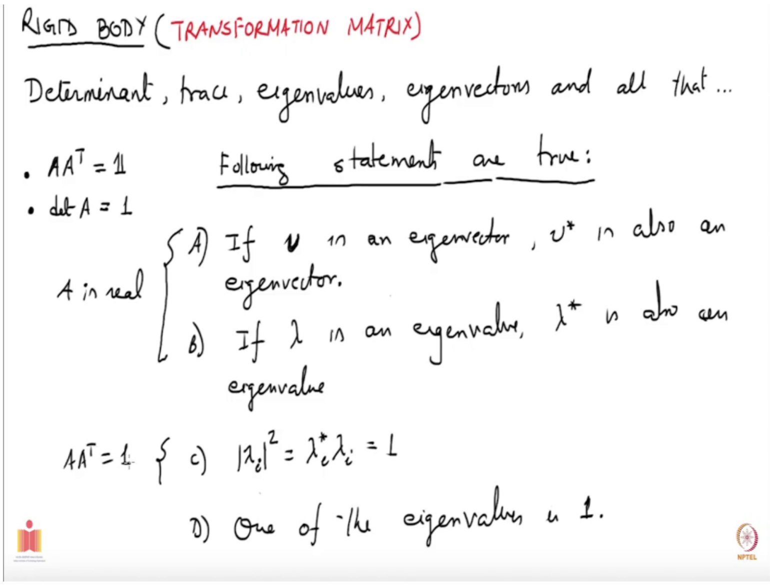


So, let us look at what we get from determinant one condition. So, I am going to use determinant of A is equal to 1. So, you take here matrix, even before I do determinant thing, let me just, I skipped a bit, jumped a bit. So, first things first. Let us say, V is an eigenvector corresponding to eigenvalue lambda. So, this is true. Then if I take complex conjugate, then I get A star, but A is real, so A star is same as A, so I get A and V star. On the right-hand side, I get lambda star V star.

So you see if V is a eigenvalue then V star is an eigenvalue with a, sorry, is an eigenvector with an eigenvalue lambda star. So, that is our claim number, let us go back, which one was it? First one, first two. A and B, I have proved now. And then, let us do the following. Let us use orthogonality. So here, I have used reality; orthogonality. So I have one relation this. Let me do a transpose of this equation. So right now, you have a matrix multiplying a column, I want to turn it into a vector multiplying a matrix.

So if I take transpose of this, it will give me V star transposed. It just means a row vector times A transpose is equal to lambda star. This is a constant, there is nothing, it is not a column or something, there is nothing to happen to this. And then, you have V star transpose. And I will take this one and multiply with this equation. So what do I get? I get V star transpose A transpose, and then you have from here Av.

Now, let look at the right-hand side. Lambda star V star transposed and I have to multiply the right-hand side of this one. So I get lambda times V. Maybe I can multiply lambda here and V V here. This is modulus of lambda square times V. Let me call it modulus of V square. Now, what is this here? This is 1 because it is orthogonal. So on the left-hand side, also you have modulus of V square. Right-hand side, you have mod lambda square V square. And from this, you can conclude that all the eigenvalues are unimodular. They all have magnitude 1. So this is my which claim this one. This follows from A transpose A is equal to 1. Okay, that is nice.

Now, let us come to the last claim. Now, determinant is 1 and if I diagonalize my vector A, then it is after diagonalization, my A will go to a diagonal vector. Let me call Ad with the subscript d. And this will have this form; lambda1, lambda2, lambda3. See this is 3 cross 3 matrix. So, it will have 3 eigenvalues.

Now, if I find out the determinant, determinant does not change under a similarity transformation and a similarity transformation takes you to its diagonal form. So the determinant of A will be the same as determinant of A diagonal, which will be lambda1 times lambda2 times lambda3.

Now, go back to this one, not here, here. This relation implies that all eigenvalues lambda i, they all have this kind of form; e to the i alpha with some alpha there. I can put a subscript alpha i or maybe I just remove this subscript. So lambda is e to the i alpha. So lambda1 will be e to the i alpha1, lambda2 will be e to the i alpha2, lambda3 will be e to the i alpha3. So in that manner.

Now, let us say lambda1, I choose to be e to the i alpha, i alpha. Then one of the two, lambda2 and lambda3 has to be e to the minus i alpha. Why? Because I have already said that if lambda is an eigenvalue, then lambda star is also an eigenvalue. So if e to the i alpha is an eigenvalue, then e to the minus i alpha also has to be an eigenvalue.

So let me call the one which has a value e to the minus i alpha as lambda2. That is what I choose to call it, and we have this relation. And my determinant of A should be 1 and I have these two, which means that lambda1 into lambda2 into lambda3 1 implies e to the i alpha, which is lambda1, e to the minus i alpha, which we have chosen for lambda2, times lambda3 is equal to 1. And from this, you can conclude that lambda3 is equal to 1.

So at least one of the eigenvalues is unity and of course, if you take alpha2 and if alpha0 is allowed then, all of them are unity, and so forth. But our interest saying that at least one of them is unity, which means the following.

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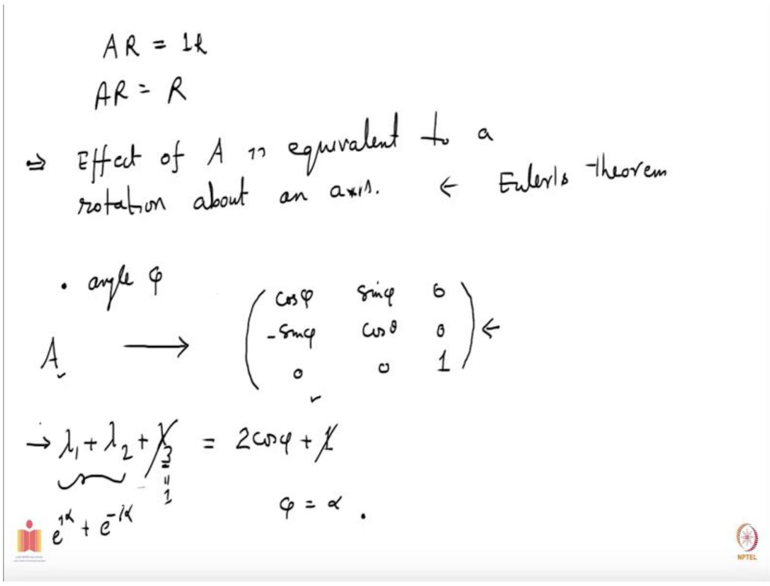


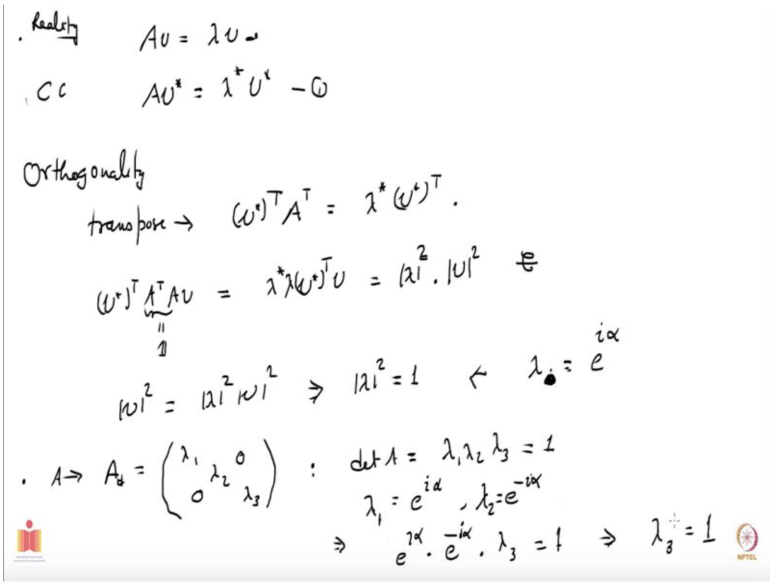


If you take your vector A, then there is one vector which I call, let us say R. So, the V, one of the Vs, I am calling R, the eigenvalue. It has unit eigenvalue. So it is basically, AV is lambda V, where lambda is 1 and V, I am calling R, just to give it a name. Or AR is R. Meaning there is a vector R, which does not change at all under the transformation. So when the matrix A or when the rigid body is moving in whatever fashion it is moving and you look at the matrix A which takes its, it takes the rigid bodies from its initial configuration to the final configuration. This equation is saying that there is a vector which does not change under all these transformation. This is one such factor.

The effect of A on R is to do nothing, which means that R should give, R is an equivalent of the direction of the rotation. You see, when you rotate a body about some axis, all the points on that axis do not move at all. All other points, which are not on that axis, they go from here to there. They rotate by some angle. But the points on the axis of rotation do not change. So that vector, that vector which is aligned along the axis of rotation that does not get change.

So I can conclude that the entire effect of matrix A can be thought of as a rotation about some axis and the direction of that axis is given by R. Is it clear? So I can conclude effect of A is equivalent to a rotation about an axis and this is called Euler's Theorem. Let me write it down. It is a nice resolute. I do not think it is obvious immediately why this should happen. I do not think it is trivial to say that in whatever fashion you have moved around your rigid body. Remember, one point is always fixed, okay? It has a fixed point. And this entire, I mean, from one orientation to another orientation is equivalent to a rotation about an axis.

So this is a nice result which also means that I should be able to do the following. You see, if you say that the entire effect is equivalent to a rotation about an axis by some angle, let us call it alpha, alpha, or phi. We will call it phi. So the entire rotation is equivalent to, the entire transformation is equivalent to a rotation by an angle phi. Or maybe capital phi probably, whatever it does not matter, angle phi. Then look at that matrix which does this rotation. It is just if you take the axis of rotation to be the Z-axis, then it will be just cos theta sin theta minus sin theta cos theta.

So if you orient your axis properly, then the matrix will be equivalent to this, cos phi sin phi. I did not mean theta, I meant phi, minus sin phi cos phi and we are saying we will choose the direction of R to be the Z-axis, then this is the matrix which does the rotation, which means that I can go from A to this one by a similarity transformation. See, this is the one which does all the job. But we are saying, whatever it does is equivalent to this rotation. So means, I should be able to find a similarity transformation from here to there, which will be equivalent to a rotation about the new Z-axis.

Okay, that is good. Now, how should I find out phi? See, I can find out the direction about which that rotation has to happen by solving the eigenvalue equation. So I find out the vector R by solving the eigenvalue equation. Now, how do I find out the angle phi? The amount of ration that is needed. And that is easy because you know that the trace does not change under the similarity transformation. So if trace does not change under a similarity transformation then lambda1 plus lambda2 plus lambda3 should be same as cos phi plus cos phi which makes it 2 cos phi plus 1. So, I have added up the diagonal entries here and these are the diagonal entries of A diagonal.

Now, your lambda3 was 1, so this cancels with this. Lambda1 and lambda2 are e to the i phi and e to the minus i phi, which makes this 2 cos phi, which means, sorry 2 cos alpha. So on the left-hand side, you have e to the i alpha plus e to the minus i alpha which is equal to 2 cos alpha which means that phi is same as alpha. Meaning when you find the eigenvalues, the alpha that you get is this phi. So that is how you find the angle of rotation also. Let us see what else I wanted to say. Okay, very good.

So I want to leave you with a question. It looks like I cannot diagonalize this. Looks like I cannot diagonalize the matrix A because I have this. It is a rotation matrix, it is not a diagonal matrix. For some time back, I said that I can diagonalize and I have this one. I had all the entries as diagonal entries somewhere, I should have written down. Here, lambda1, lambda2, lambda3, and then, you have cos phi all these things. So these are the two things and then also you realize that eigenvalues are complex, e to the i alpha and e to the minus i alpha.

I would suggest that you think for a while and be sure that you are completely sure about all these different parts fitting together nicely in your understanding that on one hand, it looks like it cannot be diagonalized because it is having this form. On other hand, you already know it is diagonal. Just make sure it fits in all nicely in your understanding. Okay, so we will continue further in the next video.