Introduction to Classical Mechanics
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Parameterization using Euler Angles

Let us continue our discussion on rigid body Kinematics in this video. In the last video, we started talking about Euler angles and as you may recall these angles are used to parameterize the orientation of the rigid body at any time t. So you imagine that you start with your body in some configuration so it is oriented in some manner in space and we attach system of coordinates to the body and the origin of this system and the system which is fixed in space coincide. So, they are at the same place.

And the question was how do I parameterize the orientation of that body system which is rigidly fixed in the body that system of coordinates with respect to the system which is fixed in space. And that is what we looking at last time and we talked about Euler angles and that is where we left and let us continue from there.

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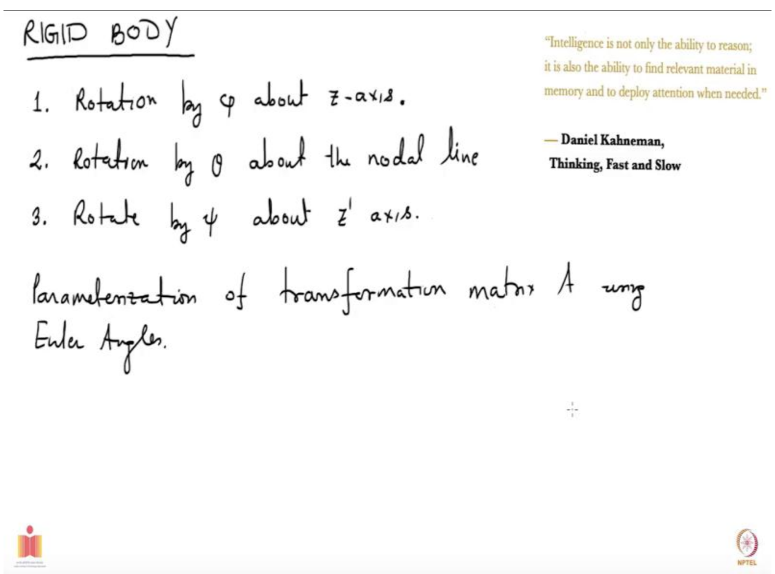

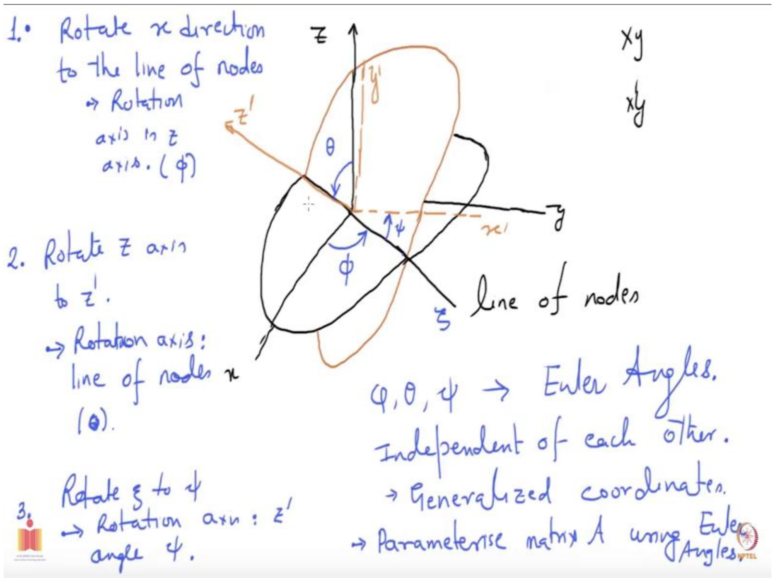

So, the first thing we noticed or I mean we did was do I mean I am describing you again the procedure these all these Euler angles. So, how do you arrive from one place to, one configuration to another. So, you start with your x y z coordinates and your body coordinates are sitting on the top of it. Then the first step was to make a rotation by amount phi about the z axis so let us remember what we did last time. So, first rotation by phi about z axis and then what we did and just to remind you this phi (amoun) this phi rotation takes you takes your x axis to the line of node that is that is the phi.

That is what characterizes phi that is what, you know the how do you choose phi you choose phi that much which will take you to the line of node. Once you are there you do a rotation and you take your z axis and rotate by an angle theta about the line of node. So, rotation by phi about z axis that one is good. So, you have rotated your, the rotation has happened in xy plane. Then second is do a rotation by theta about the nodal line. Third, so at this stage what has happened is you have already brought your z axis to its final destination that has already happened.

What has not happened is your x and y axis have not reached to their final destinations. But only one rotation will take you there and you remember you have to rotate, rotate by amount by an angle psi about what axis can you can you recall that. Let us go here, let us see, this what we are doing here, z prime. z prime is the one which is the final destination so you have already arrived at the final destination and of course you do not want to disturb that axis by doing a rotation which will involve changing this one. So, you are going to do a rotation about this z prime axis the final destination about z prime axis.

So these are only three steps involved in here and now you would like to know how your matrix A looks like when it is parameterize in terms of phi, theta and psi, these Euler angles. So, what I am looking for now is parameterization of transformation matrix A in using Euler angles using, Euler angles that is what I am going to do next. Well it is fairly, fairly easy thing to do if I understand these three steps. So, what is do is I start from my x y and z. May be I should go to the next page it will be better. Let us go to the next page.

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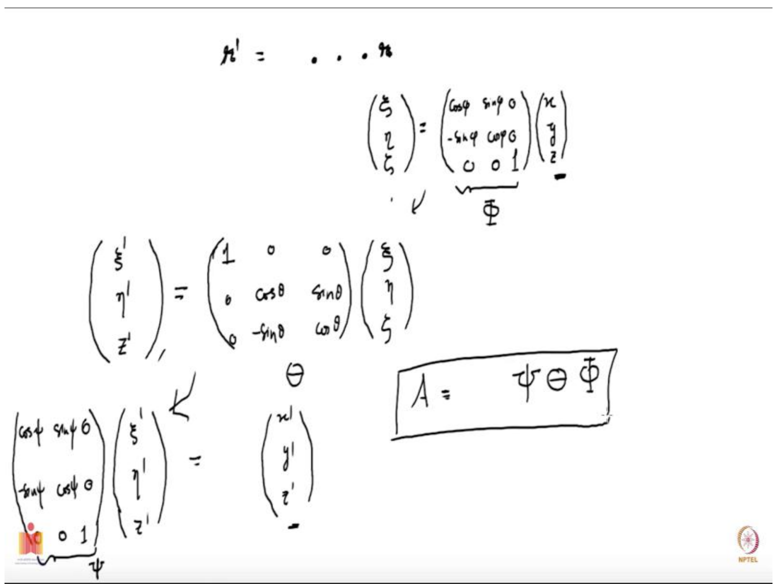

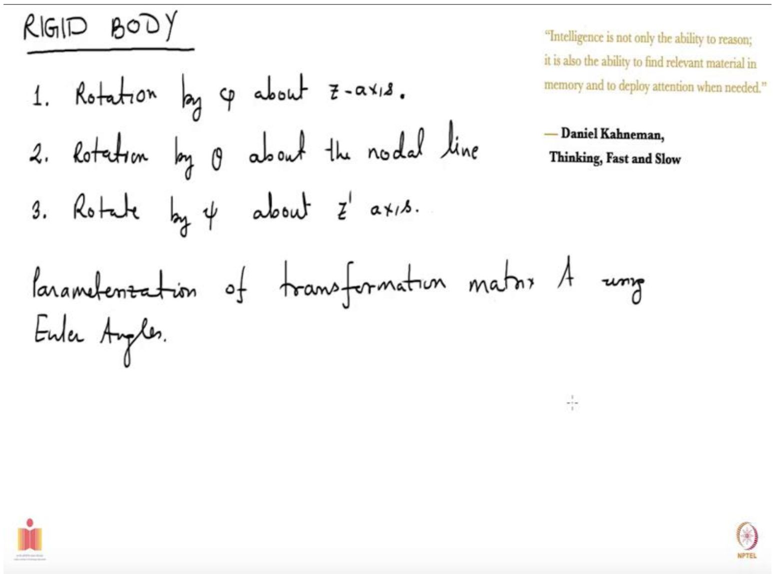

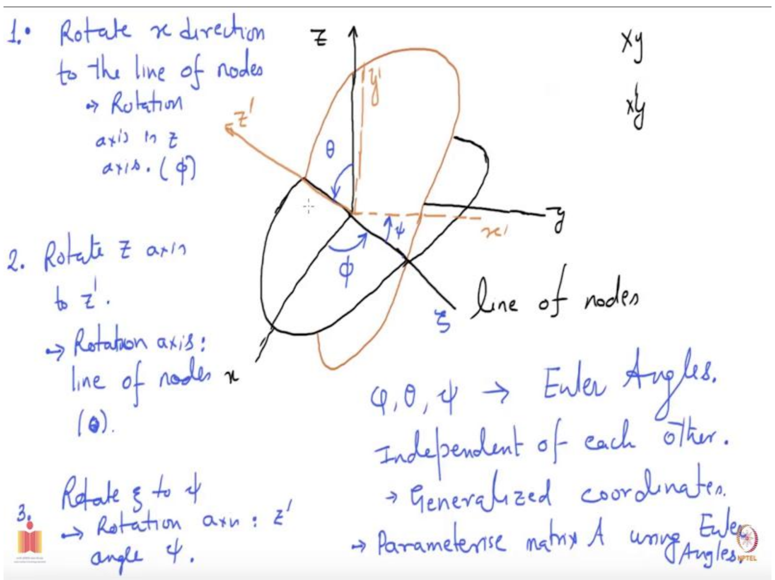

So I start with my vector which is let me call it r we are denoting it by our bold r these components x y and z. Now, I want to arrive at my final destination which is r prime. And I have to go through three sets of rotations. So, I have to have three sets of rotations here. So, what I do is I take my x y and z or z and what should I do, let us go back rotate about phi about z axis.

So, the rotation is going to happen about the z axis meaning after rotation my whatever I get it should have the same z, same value of z. And here I should have, z should not change which means these entries have to like this because this one will ensure that you, your z is not change and then you have your matrix cos of phi sign of phi minus sign of phi and cos of phi, that is the matrix. So I call this matrix as matrix phi, it is just the name of the matrix I give. And if I do so I will my x and y values will get transformed to something else and z will remain the same.

And I want to call these as zeta, eta and xi. So the first thing the, so let us say these are the components it is called xi, eta and zeta. So, clearly eta is same as z that has not changed. So, we have arrived here, so what has happened now is you are here, so this is your xi, that is where you are sitting now and other things have change accordingly. Now, I should do a rotation on this entire thing. So, now we are here now we have arrived here so xi eta and zeta and remember your zeta is z the z basically.

This is the same as zeta because I have done a rotation about it. Now, I should take I should do a rotation by angle theta about the nodal line meaning my xi should not change at all because the rotation is going to happen about it. So, let us do this rotation and I will call that rotation as the rotation matrix to be theta. So, now from here I do the following, I take it and I will take my xi, eta and zeta, rotation is about which axis? Let us go back xi, so just like what we did here, for xi not to change I have to have these as these entries.

Now, you have to mix eta and zeta and angle is theta so it has to be cos of theta, sin of theta minus sin of theta and cos of theta and I will call this matrix as matrix theta, perfect. And then third rotation and before that let me label these. Now, after this what has happened is my zeta has reached has become z prime so let me write z prime here. And let us call these intermediate values to be xi prime and eta prime.

Now, I should take my eta prime, sorry xi prime, eta prime and z prime and do a rotation, I have not left much space here which I should have done but anyway. Now, what is not going to change the z prime is not going to change so clearly here you have to have 0 1 and 0 0 1. And what is the angle of rotation? Rotation angle is psi so the rotation, this rotation is going to happen here and let me so write down cos of psi sin of psi minus sin of psi cos of psi and I will call this matrix as matrix psi.

And because there is no space left on the left, left on the left is nice, so I will write it on the right hand side. So, I wanted to have flow like this but it did not work out but anyway does not matter. So, I get where x prime y prime and z prime. So, I have finally achieved what I set out for. So, I start from here and arrive here through three consecutive rotations and these are the corresponding rotation matrices.

Which means my matrix A which takes me from original configuration to the final configuration is a product of these three and because this is the first transformation I should put it to the right most because that is one (multi) that is the one which is multiplying my x y z and then I have done by theta and then I am I have done by psi, so this is what my transformation matrix A is.

So, it is straight forward to now determine what this matrix A is by just multiplying these three matrices. I will give you the answer for that and you should multiply that to see whether you also get the right answer. Let me write it down.

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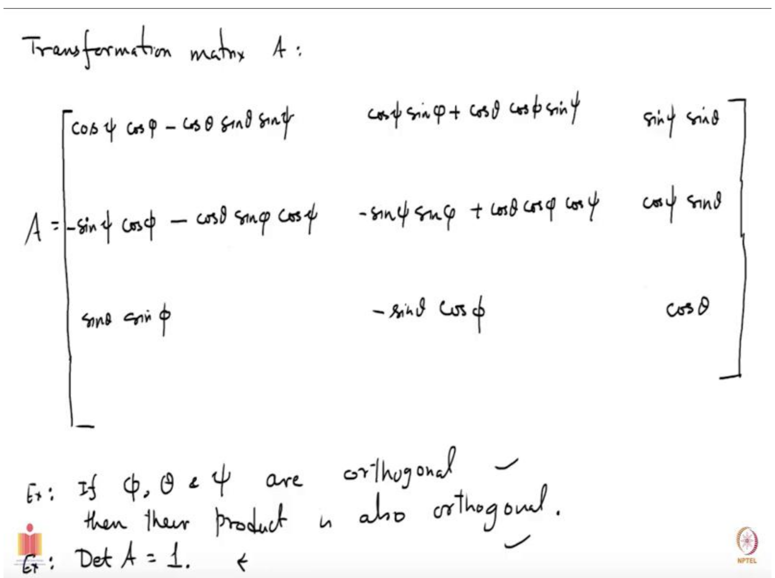

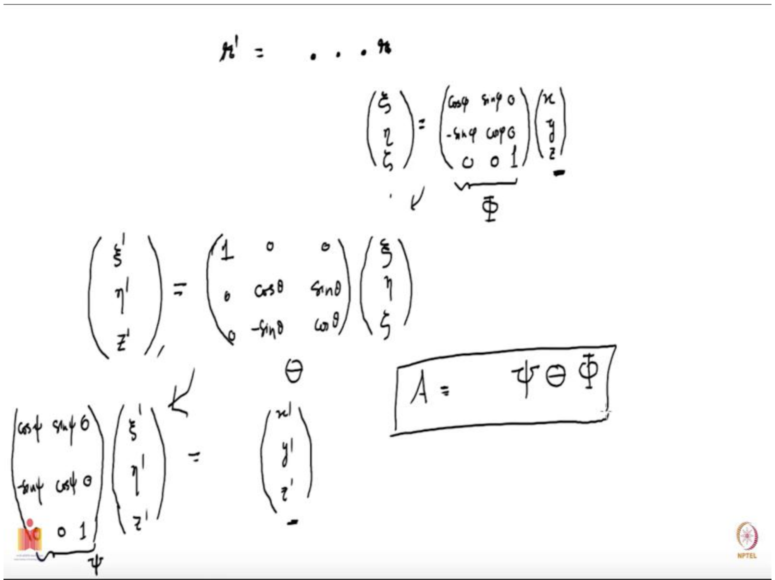

So my matrix A parameterized in Euler angles says the following, it takes a while to write this let me see. I will put the brackets later now. So, the answer if I have made no mistakes in deriving this should be the following, cos of psi times cos of phi minus cos of theta sin of theta sin of psi, it will be good if you can use Mathematica to do this multiplication.

See it is a trivial multiplication and you can of course do by hand but you can also after you have done that let us say you do it in Mathematica and verify your answer so keep getting practice with Mathematica. That is good, there is the first entry then you have minus sin of psi cos of phi cos of theta sin of phi cos of psi. While I am writing these entries what you can do is try to see whether these entries appear to be correct.

So, you may try to look at certain limits and see whether the entries make sense, for example, you could put theta phi and psi to be 0, all of them and try to see whether what I am writing gives you an identity matrix. So, for example, while I am writing all this you can check whether this becomes unity in the limit psi, phi and theta to be 0. So, and corresponding to other entries are 0 or not. So you may keep checking while I am writing this.

Where is it? Second term is let me write the last term here. They should be sin of theta sin of phi then here is cos of psi sin of phi plus cos of theta cos of phi sin of psi and then you have minus sin of psi then sin of phi plus cos of theta cos of phi cos of psi. Let me look at this one whether I get 1, there is minus sign so I am curious. So, if phi and psi are 0 this term is gone. So, minus is not creating any trouble that is 0. And these are all cos which means they are 1 in this limit so it gives you 1 and in this diagonal entry so it looks good.

So, let us look at this one, sin psi sin of theta cos of psi sin of theta, let us check this one because theta 0 will give you 0, it is good. This anyway you expect to be 0 this term and this term and this term and this term, they all should be 0. Let us write here minus sin of theta cos of phi and then the final entry is cos of theta which will become 1 when theta is 0. So, in that limit you will have all the entries to be 1 on the diagonal and all others 0 so which looks at least good from this from this check.

And you should also try to see, see if you can come up with other checks which you can do on this. So, this practice should always be there of, doing some small checks quickly here and there. So that you are sure that everything is good. That is nice we have now matrix for parameterizing different orientations of our rigid body with one point fixed. Remember we have fixed one point but if that point is not fixed it is not a big deal because you can just move that point around just by a translation so that is not a not an issue.

Now, you may try to think about what are the properties of this matrix. So, as always the moment we see a matrix we want to talk we want to ask about what is what its determinant is are, what are its traces. These are things which should immediately pop up in our head. And you may also wonder whether this is an orthogonal matrix. After all we have multiplied 3 orthogonal matrices, these matrices theta, phi and psi, they are orthogonal and you may wonder whether the product is orthogonal.

Well we expect them to be orthogonal because any two points, the distance between any two points in the rigid body is not going to change under this transformation. So, it better be an orthogonal transformation. So, even though it is expected but nevertheless check that this matrix is orthogonal and you can do so by doing an exercise instead of checking whether this is orthogonal, directly you can also check if you wish by checking whether phi, theta and psi or better still you check if phi, theta and psi are matrices which are orthogonal then there product is also orthogonal.

And then we would like to know about the determinant. Well again let us say you have shown these are orthogonal then there are two possibilities, plus 1 and minus 1 for the determinant. And you expect that determinant to be plus 1 that because remember it is the determinant plus 1 part which is connected to the identity in continuous manner. So, because you can build up your matrix A by doing infinitesimal transformations, you expect your determinant of A to be unity but it will be nice if you can explicitly verify that.

And again you can check explicitly by taking this whole matrix that that determinant is 1 and also you can show that if you have a product of three orthogonal matrices or any number with which are orthogonal and there determinants are 1, the product will also give you determinant 1. So, that is what you can you can check. I think this is probably all that I want to say in this video. We will continue further in the in the next one.

I will encourage you however to keep coming up with your own questions which you can turn into exercises for yourselves and solve it and Mathematica, using Mathematica will be a very useful way of doing. So you do not keep multiplying like all these numbers again and again, a computer can do it for you. But such proofs, this proof, these kind of questions you can come up and do that algebra to see whether you are able to you know satisfy your expectations from these, these question. So, see you in the next video then.