Introduction to Classical Mechanics
Professor Doctor Anurag Tripathi
Assistant Professor
Indian Institute of Technology Hyderabad
Rigid Body Euler Angles

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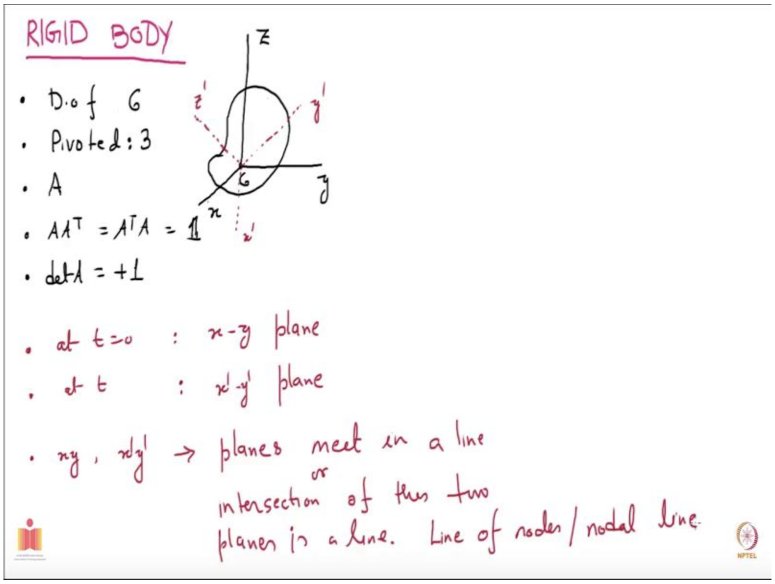

Let us continue with our discussion on rigid body. So, till now we have shown that the degrees of freedom of a general digit body is 6 and if it is pivoted it is 3. If there is a fixed point, that is what I mean by pivoted and then last time, in the last video we were talking about matrix A, which takes you from some initial configuration of the body to a final configuration. So, let us say at some time t equal to 0, the body is oriented in some manner.

Let us say it is oriented like this and at a later time, it is oriented in a different manner it turns around go somewhere else and going from here to that new orientation is carried by some matrix A and that is this matrix, we were able to show that the matrix A should be an orthogonal matrix, meaning A A transpose and A transpose A should be identity, which just means that the transpose matrix is the inverse of the matrix.

And we also saw that orthogonality implies the determinant of A could be plus minus 1, but then we fix that it has to be plus 1 by arguing that this has to be connected to the identity in a continuous manner. That is what we talked last time and from here, we want to take it further.

Now I am interested in not all the 6 degrees of freedom, I want to look at a pivoted system meaning a body which has a fixed point, let us say this point is fixed, it does not move around it, it always remains there and it is not really some loss of some generality, because you can always move this point to somewhere else and you will recover all your degrees of freedom, so 6 degrees of freedom. So, we want to discuss about the motion with 1 point fixed and we will not lose any anything.

So, let us say this point is fixed and I want to know, if this body at some other time is oriented in a different way, this point still remaining there, then I would like to know what transformation matrix A will take me from this original orientation to that final orientation. So, that is what I should be, I want to find out, but before I do that, I should be able to parameterize the orientation of this rigid body, in some way, meaning I should be able to find independent variables, which can tell how the body is oriented, unless I am able to do so I cannot, I cannot find the matrix A.

So, that is the goal of this video, we want to find that parameterization and what I want to find is independent coordinates. So clearly, because I have pivoted system, I have 3 degrees of freedom. So, I am going to find out eventually, 3 independent coordinates and that is what this lecture is about.

So, what I do is, I take this rigid body and I fix a system here, a system of coordinates here and I call this x, y and z. I also attach another coordinate system here with the same origin. But that new coordinate system will move with the body. So, understand there are two sets. One is this set this set, which I have drawn already here.

This is not going to move when the body moves. It is always going to stay fixed here at this place. But then there is another system which I have attached with the same origin here, which will move when the body moves and you can imagine sticking some rods into the body rigidly, so that when body moves, that coordinate system also moves.

So that is what I am going to do and let me draw it, draw that body system at a later point of time. So, let us say, let me draw it with a different colour, something more happier. So, let us say at some another point of time t, I have my x here, y somewhere else, and z somewhere else and I will call these as prime coordinates, x prime y prime and z prime and what we really want to know is, what matrix A is going to take us from our space coordinates, when I am saying space coordinates, I mean, this rigidly fixed system to the body coordinates, to this system.

So really, the transformation is the following. You tell me those three numbers, with which I will know what I should do with these body coordinates, which at time t equal to 0 coincide with the space coordinates. So, with those three numbers, I should be able to take this x and put it at x prime, take y and put it y prime take z and put it z prime, if I can do so then I will be able to exactly tell the orientation of the body.

So that is the goal. I hope as we go along, it will be even more clear what I am trying to convey. But I still believe that it is what we are after is clear. So, let us see, so here let us say this is my original xy plane, meaning this x and y plane. So, I want to, I want to draw this xy plane and after the transformation has happened and you have got new, you have new x, y prime z prime orientation of the body, you have new plane, which is x prime y prime plane.

So, we have two planes here, to begin with, let me go back, to begin with at t equal to 0. I have these xyz and I am right now, for some reason, which I will tell you interested in the xy plane, at time t my body is oriented differently and I am looking at x prime y prime plane. You will see why I am interested in that.

Now two planes meet in a line. If you, if you take two plane planes, they will intersect somewhere and that intersection is a line. So, if I draw both the planes, now xy plane and y, x prime y prime plane, they will intersect somewhere and these planes meet in a line or the intersection of the two planes is a line, of these two planes is a line. That is nothing profound I am saying that is something which you have known for ages.

But this line is going to be very useful for us. So, I want to give a name to it before I proceed and I want to call this as in line of nodes or nodal line, either way, I will address it. So, very good. There is a line of course, I call it nodal line and now my question is, how do I start from x y z and arrive at x prime y prime z prime and that is what we are going to look next.

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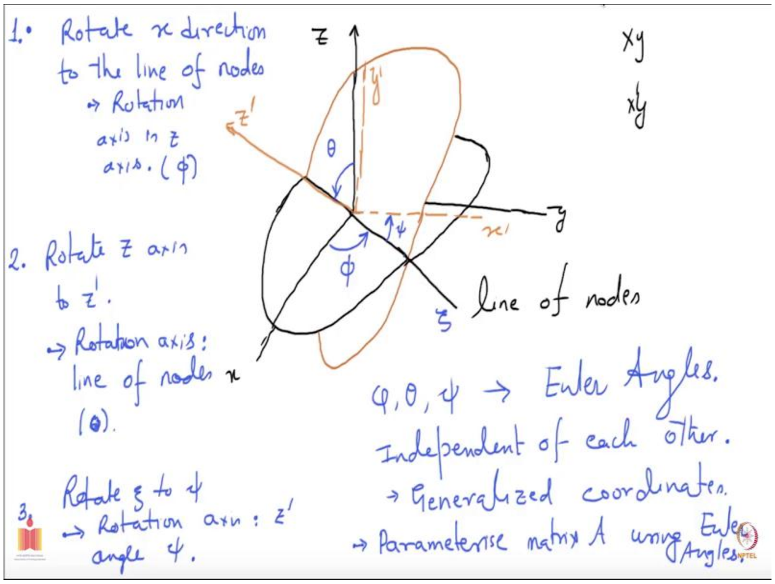

Let me make it black. So, let me first draw the x y plane and x y, y prime plane and show the intersection. So, let us say this is the line where they are going to intersect the two planes. This is my xy plane and then I will draw my x prime y prime plane, maybe I should use a different colour, yeah, I think I should this is my x prime not looking very nice, something like this and of course, I can now complete my xy plane, so that it looks easy to understand.

So, this is the plane which is going I wish there was more inclination. So, this is the black thing is xy plane, the golden thing is x prime y prime plane. So, x y and x prime y prime plane and this is your line of nodes this line. Let me draw my x, y and z, let us say here, here is your origin and this is x axis, this is y axis and here will be your z axis perpendicular to the xy plane, perfect.

Now let us look at x prime y prime plane. Let us say, let me use golden, let us say this is my x prime somewhere here would be my y prime, these two axes are within this this plane, this golden colour plane and then the z prime will be perpendicular to them, perpendicular to that plane. So, it will go something like z prime.

Let me change the colour now. Let us see, very good. So, let us say I start with x axis and I want to bring it to x prime. I want to bring y to y prime and bring z to z prime, that is my goal and I will do it in steps. Now instead of trying to bring it to x prime, which is not so easy to do at first, because you do not know where it is, what you do is, you say of course, I do not know, I mean I know the line of nodes. So, I know where the planes are intersecting.

So, what I will do is, I will take my x axis and rotate it to the line of node that I will do and that angle I will call phi. So, you rotate by amount phi and arrive at line of nodes. So first, transformation is rotate, I do not know what I said, just now, in case I said something wrong, let me let me state it again. I think everything I have said is correct.

So, I rotate x axis to the line of nodes. Correct I think I have said everything correctly. Rotate x axis, x direction to the line of nodes or nodal line. That is fine, but nothing has reached x has not yet reached x prime, I will do that later. But then yeah, before I tell you what I do next, I will ask you, can you tell me about which axis this rotation has happened.

So, I have rotated by amount phi about which axis, while the rotation has happened in the xy plane, of course, because you are still in the xy plane. So, this has happened in the xy plane, which means that the rotation axis is z axis. So here, for this, the rotation axis is z axis, is z axis. So, that was my first step and angle was phi.

So, now my new x axis, x new I mean the x has moved here, it is along the line, y has of course, moved by another I mean amount phi further in this direction and z nothing has happened and z prime, z is here and z prime is there. Now, what you do is you take your z axis, which has not moved at all, because the rotation was about this axis. Now, you take that z axis and rotate it into the prime. So, you take it and rotate it into z prime, you had to bring it here and let us say the angle is theta. So, I want to denote the angle I need to rotate it into that prime by theta.

So, second step is rotate z axis to z prime. Now about which axis this rotation has happened, can you quickly see, rotation axis. See it has gone from here to there and both the z and the z prime, they are both perpendicular to the line of nodes, is it clear. These both these directions z and z prime are perpendicular to the line of nodes, which means the rotation is happening about the line of nodes. So, this is, this is the axis the line of nodes, you may have to visualize is it by taking some let us say pencil or some sticks and try to do it, it is slightly difficult to show it in planar diagrams.

But anyway, I hope it is clear that the rotation axis is line of nodes and we have rotated by amount theta. So, theta good. Now, what is the situation? The situation is this axis has come here, z axis has come to z prime meaning its final destination. So, this as far as z axis is concerned it is at the right place, as far as x axis is concerned it is here but it should have been here and y axis is somewhere else. Now, what I do is, I take my x axis which is here lying on the line of nodes and rotate it by angle psi in this plane, this plane, the x prime y prime plane, this plane I rotated.

And when I do it, my x axis would have moved to x prime my z axis is anyway sitting at the right place already and because there is no other possibility for y, it has to be coinciding with the right y prime. Because there is nothing else where y can be, it has to fit on the top of y prime. So, you do a rotation, rotate your this one, this the new one let me call it I think I call it xi or something I will correct it, if this is not the same notation I am using. But anyway, for the time being let us call it xi and I will rotate it to psi.

So, rotate xi to psi if you do so, then everything is in the right place. Now, let us ask again what is the rotation about for this psi, can you think of the axis about which the rotation has happened which has taken you from here to there, well you see your rotation is happening in this golden plane, in the in this plane and clearly your z prime is perpendicular to it. So, your rotation is happening about z prime.

So, your rotation axis is z prime and the angle I call psi. I hope it was easy to understand, but if it is not, it is not because it is difficult, it is just because it is in 3 dimensions and you need to sit down and try to visualize 3D. The difficulty lies in it being in 3D rather than any other intrinsic difficulty.

So, anyhow, we have now a way to parameterize the orientations. So, if someone asks us I mean someone takes a top and puts it in a different orientation, I can tell how it is oriented by telling these three numbers phi, theta and psi and if I tell him these three numbers, then he will be able to orientate in, in that fashion. In a sense, so let us say a top is oriented in some way and I want to tell him or her to orient the top in a particular way.

So, all I need to tell him that please rotate by phi about z axis and then you rotate about line of nodes by amount theta and then you do a rotation of the line of nodes, because right now, it is the line of nodes your x is sitting on the line of nodes by some amount psi, about the z prime axis and if at all tell all these three things, he or she will be able to arrive at the configuration at which I want it to be.

So, this is the way I can specify the orientation of a rigid body whose one point is fixed. These angles are called Euler Angles or Eulerian Angles whichever way you want to call them. So, theta, phi and psi, these are called Euler angles and these are independent of each other, which means I can use them as generalized coordinates. Remember for writing down Euler Lagrange equations, we need generalized coordinates, we need coordinates which are independent. So, these three constitute a set which can be used for generalized coordinates

So I can use them as generalized. So, now that we know what these Euler angles are and how we have to go from one orientation to the other orientation of a body, using these Euler angles, I should be able to write now the matrix A. So, that is what I will do next in the video, goal will be to write the matrix A using Euler angles and I will be carefully labelling all the axis and their intermediate steps and we will arrive at the parameterization of the matrix A.

So, the next goal is to parameterize matrix A using Euler angles. My experience in classroom tells that many students face difficulty with this and usually they will be confused with the axis about which these particular, these specific rotations are happening. So, if you pay attention and spend some time this will be very clear, it is not difficult thing, there is no intrinsic difficulty in understanding this. So let us see in the next video, how to parameterize this matrix. See you in the next video then.