Introduction to Classical Mechanics
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Rigid Body, Transformation matrix

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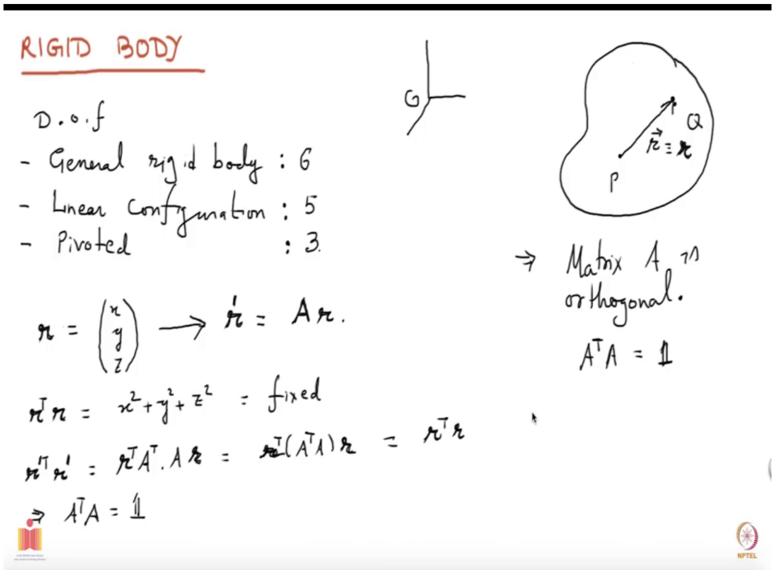

In the last video we started talking about Rigid Bodies and we defined first what a rigid body is, it is an object whose points, the points which make that object. The distances between those points are fixed, they do not change with time that is the definition of it and it is a good approximation if you are not interested in details of motion that happens within the object.

Then we looked at the degrees of freedom it has and in general a rigid body has 6 degrees of freedom that is what we evaluated and while doing so, we first encountered linear configuration. So, if you have a rigid body, which has a linear shape, which is like line, then it has 5 degrees of freedom that is what we saw and if your rigid body is pivoted at some point, meaning it, if it has a fixed point then it has 3 degrees of freedom that is what we saw last time.

Let us continue further and let us imagine that we have some rigid body. Let us say this is a rigid body and we have our coordinate system here. So, these are our X Y and Z axis and this is the origin and think of any two points in here, let us say I call this point as P, that point as Q and let us say the vector which connects the points P and Q, this let me put it put the Q there.

This vector I want to call as vector r, so from here to there. Now let us say this body is moving as time goes on. So, this vector r will be changing with time. So, the vector which connects these two points will be changing. So, right now it is in this direction and has this magnitude and of course the magnitude is not going to change with time, but its orientation will change, it may orient at a later time in this direction or the vector r could come out of the, of this plane and be pointing towards you. So those kind of things will happen.

Now I want to view this change in vector r as a transformation, as a linear transformation, the transformations will be linear. Let us see what precisely I want to say. So, sometimes I will not use this vector arrow on r and try to see if I can do a bold face, is it easy? Yeah, it is easy, it is doable. So I will use a bold face and this is the same thing here, same vector which I have denoted there, let me also.

So, by r I mean, its x y and z components that is what I mean and I am viewing it as a column vector right now and as time changes this vector r will become something else and it will become some r prime and that r prime will be related to your original vector r, this original column vector r and there will be a matrix, which will be multiplying that.

Just say, see matrix multiplication is a linear operation and that is why you have it here. So, this is how your r is going to transform, so your r is going to become r prime which is A times r and A is some matrix which we want to figure out. So, let us do the very first thing which we would like to do is let us say look at r prime or let us say r transpose r, r transpose r.

Now r transpose r is just x square plus y square plus z square and that is the square of the distance between these two points and that point, that number that quantity is not going to change with time. Because this is just the distance between these two points and it is a rigid body, it is cannot change with time. So, this is going to remain fixed as time goes on as far as these two points are concerned or any two points are concerned.

But now we are saying this vector r is going to change into A r after some time. Then what is my r prime transpose r prime. Well, this will be r prime transpose would be, I have to take the transpose of this quantity, which is r transpose. Then you have A transpose, A is a matrix that is good, then you have this piece, r prime is A times r. So you get r transpose, let me put a bracket around this just to emphasize that I am looking at this quantity, you have this quantity.

And as you know, this has to be equal to r transpose r because the lengths are not going to change with time, the distances are not going to change with time between these two points. Which means that A transpose A has to be identity matrix. So, that is the condition which we have found by asking that the distance between these two points should not change the points P and Q and it says about the matrix A1 property of this matrix A that it A transpose A should be identity and such matrices are called orthogonal matrices.

So, which means Matrix A is orthogonal. That is the definition of orthogonal matrices that they transpose. So, if you have a matrix A and if it satisfies this it means that its transpose is its inverse, that is what it means. So, the transpose and the matrix are inverse of each other. That is what we have found and one very important property of a matrix is its determinant.

So, whenever you come across a matrix, you think about it stress, you think about its eigen values, you think about its determinants, these are the things which we should pop up in our head. So, let us ask about its determinant, if it is an orthogonal matrix. So, let us ask the determinant of Matrix A, if it is orthogonal. Let us go to the next page.

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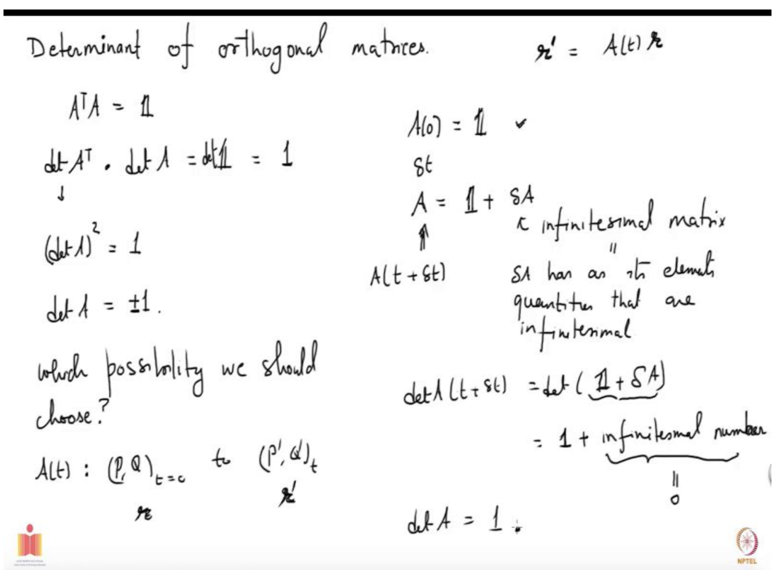

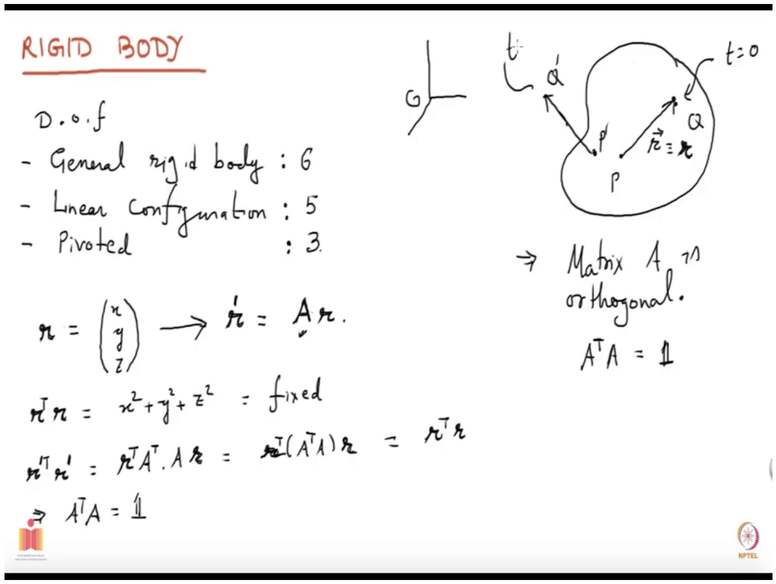

So what we want to know is determinant of orthogonal matrices. Let us see. So write it down, you have A transpose A as identity. Now, let us take determinant of both the sides which means determinant of A transpose times determinant of A is equal to identity. That is the property of determinants, so product of two matrices becomes product of their individual determinants and on the right hand side also I should have determinant of this and determinant of identity is just 1 that you are multiplying all the diagonal entries in this case, it is 1.

Now, it is also true that determinant of A transpose. Let me write it here, anyway, so determinant of A transpose is same as determinant of A, so that is what I am going to use. So, determinant of A times again determinant of A which makes it determinant of A square and that is equal to 1. This 1 is a number not a matrix, which means our determinant of A could be either plus 1 or minus 1. So, there are two possibilities for our matrix A and it is that the determinant could be either plus 1 or minus 1.

That is nice. Now out of these two allowed possibilities. Which one is true for our case? So, our case meaning for the rigid body. You see at this moment, I have found out that if you are taking any orthogonal matrix its determinant could be either plus 1 or minus 1. But it is not necessarily true that whatever possibilities appear here, apply to our case because we are looking at a very specific thing, a body which is moving in time.

Now I want to, before I said that let me write which possibility we should choose? Should it be plus 1? Should it be minus 1? Or both can be here? That is what we are asking. So it is not difficult to understand what we should do and it is simple to understand, if we do it the following way. Now, we can go from an initial location of points P and Q. So, we start with our body in some configuration point P and Q are given the way they are and we ask and we and we ask what is the matrix A that takes you to its final configuration? Let us say it is called A prime B prime, not A prime, P prime Q prime.

So let us say we start from here. Let us say time t equal to 0 you are here and after some time t let us say this thing is oriented like this. So, the point P has moved here the point Q has moved there and their new locations are called P prime and Q prime which means the body got displaced and got oriented differently that is what it is saying. So, it is here and we want to know what is the matrix A that has taken as from here to there.

So, let us say this orientation was at time t equal to 0 and this is orientation at some another time T and that is what we are asking, the matrix A of T which we will take us from r meaning, let me from P Q at time t 0 to P prime Q prime at some other time T or equivalently, we are basically asking what is, this is the distance between the two points here is r prime and here is r and what we are asking is, how do you go from r prime to r.

This matrix will depend on time because the object is going to move in time with a different times, it will have different orientations and locations and this is, this matrix say whose determinants are in principle allowed to be plus 1 and minus 1. Now how do we fix this, how do we fix our choice? So it is easy you imagine that you reach from your initial configuration to final configuration, which is given by A of T in steps, in infinitesimal steps. So, you do not go you do not take your object and turn the way we want to orient it, you make very-very small transformations, first step, second step and third step and a huge number of them and then slowly arrive at that configuration.

So, just like going from point A to point B, I can say I jump from here to there or you make small transformations and you reach at point B. No if I do so then I am, yeah, let us say I take the first, I do a first infinitesimal transformation. So, I do not reach to the final configuration. But I start from identity at time t equal to 0 r prime is same as r. The nothing has changed which means that A at time 0 is 1 is identity, is that correct, matrix A at time t equal to 0 is identity matrix because in only in that case r prime will be r. No transformation has been done.

So this is a time t equal to 0. Now think of a, of an infinitesimal amount of time delta t which has passed and your A is now it has to be identity, no transformation at all, plus a small change because of the small transformation that happens in time delta T. So, here you will have in addition to identity a little bit extra and what that extra I will write as I do not think I want to use a symbol, I want to use a symbol. I will write some delta A here and this delta A is an infinitesimal matrix.

Is that clear that this is the thing which we should write. So, my Matrix A differs by an infinitesimal amount from unity and what do I mean by infinitesimal matrix. This means that the matrix delta A has it is has as its elements, quantities that are infinitesimal. That is what an infinitesimal matrix would mean, that is good.

Now, let us look at the determinant of the matrix A, after a time delta t, so I have done a little bit of transformation, I have not reached where I wanted to but I have taken the first infinitesimal step and I want to ask what is the determinant now of this new matrix. Let me, let it be A, but if you are having any confusions, you can think of it in more concrete terms like this.

This is what this symbol A means here. So, determinant of this would be determinant of identity plus delta A, is that is that clear? All I am doing is just taking determinant on both the sides in this equation. Now, let us look at the right hand side, if the delta is not there, the determinant of identity Matrix is 1.

Now because I have identity plus an infinitesimal Matrix, then the determinant of this entire thing can differ or should be 1 plus a small quantity, 1 plus something infinitesimal and because if this is not there, if this delta is not there then the determinant is 1, you change the matrix a little bit from away from 1, then the determinant should change at most a little bit away from 1. So you should get something like 1 plus at most this can happen. So plus, so it should be 1 plus some infinitesimal number which will be determined by the transformation delta A, that is good and then you can do another transformation and again do this.

So, each time you do it your value from, the determinant value will change, can at most change by an infinitesimal amount. So you keep changing by infinitesimal amounts. But then you realize that no that is not possible. Because the only two possibilities that are available to you are either plus 1 or minus 1, so you cannot have 1 plus 0.0001 that is not allowed. So, the only way it was going to work or this will be consistent is if this is 0.

Meaning the determinant never changes away from 1. It always remains 1, then it is fine. But if you say no it can change a little bit, then you are in trouble because you are not allowed to your determinant A can be either plus 1 or minus 1 you see what you are really seeing is because your transformations are connected to the identity matrix, remember at time t equal to 0 you are here, you are connected, you are continuously moving away from identity and because of this continuous movement you are going in small steps.

Because of that thing you are finding that my determinant A has to be plus 1, for this rigid body motion. But in general orthogonal matrices could also have determinant minus 1 and we will not talk about determinant minus 1 possibility at this point of time, it is not needed. So, we have concluded that for our things, we are going to use determinant A to B plus 1.

So, make sure that you are able to understand and argument is fairly simple. But in my experience when I am telling him this in the classroom, I have to several times repeat this and students have to, students usually ask but it is much simpler than it sounds at first. So, good and then we will move on to the next parts of this discussion in the next video.