**Introduction to Classical Mechanics**
**Professor Dr. Anurag Tripathi**
**Assistant professor**
**Indian Institute of Technology, Hyderabad**
**Lecture 36**
**Apsidal Distances, Eccentricity of Orbits**

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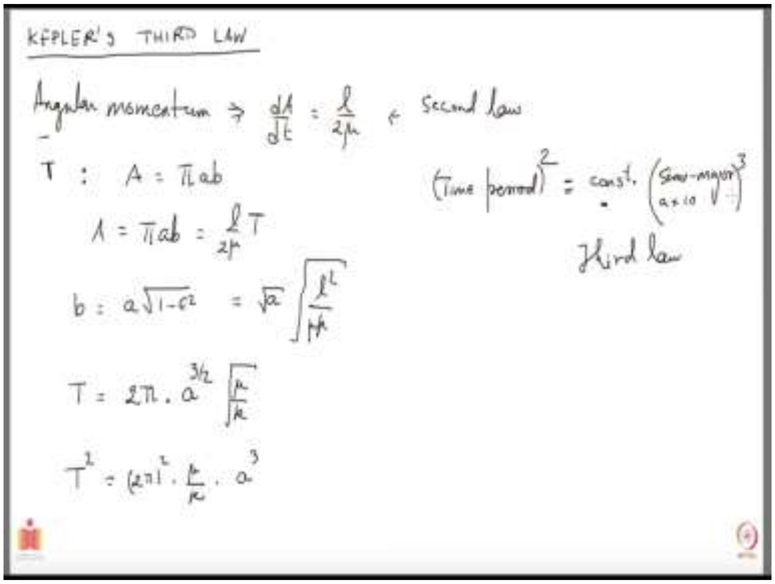


Let us, begin by driving Keplers Third Law in this lecture. As you may recall from angular momentum conservation we concluded the Keplers second law which says that the radius vector sweeps equal areas in equal times. So that is what was second law and let me write it down angular momentum implies that dA over dt is a constant and that turned out to be l over 2 mu this is what your second law is.

Now, if let us say we are looking at ellipse elliptical orbits then if it takes a time T which I write this capital T to complete one orbit, then in the time T the radius vector is going to sweep an area A which is equal to the area of ellipse that is Pi times a times b, a and b are semi-minor and semi-major axis there is the area of ellipse.

So, in this time T this much of area will be covered and if you use your second law here and integrate this equation, you can get you will get A is equal to and I am integrating this one and I am just writing what value it is. So l over 2 mu times T, I am just done an integral. Now, you recall how semi-major axis sorry semi-minor axis is related to semi-major axis and the relation if you recall is this, and if you substitute the values you will get a square root of a l square over mu k in the square root.

Now, if you substitute a, a is anywhere here if you substitute b in this expression will get the following. You will get that T is equal to 2 Pi the algebra is simple you have to just plug in few quantities here a to the 3 half mu over k in the square root which means you are T square if you square both sides 2 Pi square and you have a mu over k times a cube which says that time period square is equal to a constant times semi-major axis cube and this is your third law.

So, for given two body system the mu is fixed and so this piece is truly a constant and this is how you are time periods are related to semi-major axis cubes and that is what is the third law is. Next I want to talk about a special vector that is conserved when you are looking at Kepler problem and this is called Laplace Runge Lenz vector and that is what I am going to look at now.

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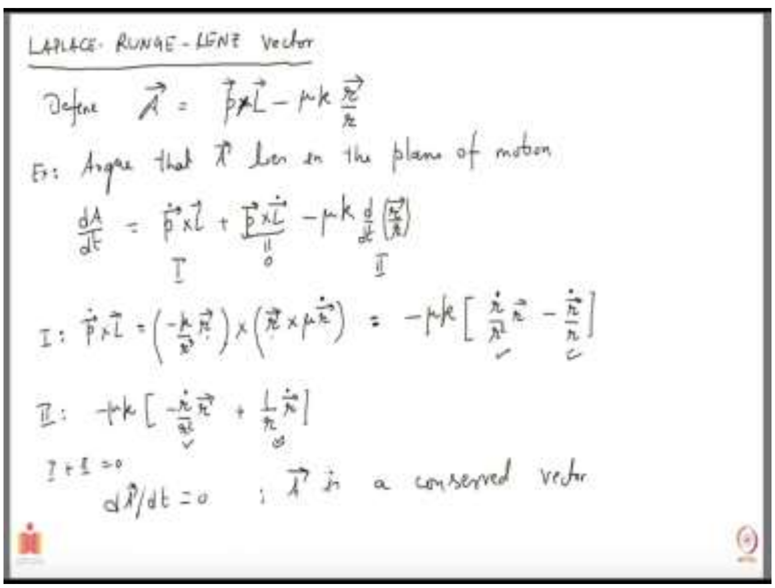


So, we will talk about Laplace Runge Lenz vector. So, out of the momentum, angular momentum and the radial vector the radius vector which gives the location of the particle. Let us, define a quantity which I denote by A as follows. So, I take the cross product of the momentum of the particle mu wherever it is with the angular momentum which is conserved. This is a cross product and from this you subtract mu k times the unit vector unit radius vector so I have divided by r which makes this quantity a unit vector.

A small exercise trivial one basically argue that A lies in the plane of motion. So, you should argue that A the victory A is a vector which is in the plane in which the particle is moving that is fairly simple exercise. Now, I want to show that this is a conserved quantity meaning I want to show that if I take the time derivative of A, I will get 0 that is what I want to show.

So, let us calculate the derivative. So, I have p cross l, so first p dot cross l plus p cross l dot that is good minus mu k, I have dr over sorry d over dt of this I will save it for later. Now, this is 0 immediately. Why? Because l is conserved, so d over dt is 0 and that is what is here. You have l dot so it that is 0 so this is gone and let us now look at these two terms 1 and 2.

So, term 1 is a following so I have p dot cross l. Now, p dot is the force, the rate of change of angular of momentum is force and our force is minus k over r square in the radial direction which I will write as minus k over r cube and I write r here which makes it go as 1 over r square and the direction is captured correctly here.

So, you have p dot now let us multiply with this across with this angular momentum and angular momentum is r cross p which I write as r cross p would be mu r dot that is good. This is same as maybe I will leave it as an exercise. See you have here r cross r cross r dot and this I will just write down the result it is simple algebra you can do.

So the result will be minus mu k then you have r dot so there is no vector here this is just r dot the radial velocity over r square times r vector minus r dot over r it should be not very difficult to obtain then the term number 2, you can check it will has simple derivative so you have you will get minus mu k and then you will have minus r dot over r square times r vector plus 1 over r r dot.

Now, let us see what is sum of 1 and 2 so you minus have mu k these are common here in both. So, you have r dot over r square r, r dot over r square r with a minus sign so that cancels against this one and similarly the second one also cancels. So 1 plus 2 is 0 which means dA over dt is also 0 which would imply that A is a conserved vector, and this is Laplace Runge Lenz vector is.

Now this is a conserved quantity so should you be expecting a symmetry which will lead to this conservation I will leave you with that thought about this. Now, what you can ask about this vector A. Now, you may think what is the magnitude of this vector A and where is this vector located ofcourse it is going to be fixed in space and not change with time because this is conserved and you would like to know where that vector A is located and what is its length that is what we want to take up next.

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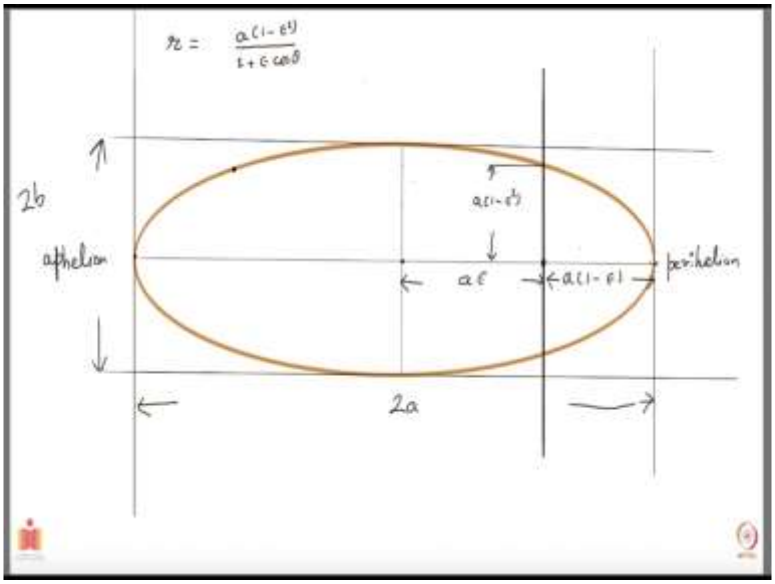


So, it will be very useful to mark all the quantities which are related to ellipse before I calculate the magnitude and direction of that Laplace Runge Lenz vector. So, I have drawn an ellipse here and let me just fill some of the details here is the focus this point is called perihelion so some nomenclature I want to throw, perihelion this point the opposite point which is furthest away from the focus is called aphelion.

Let me also write down the equation which we wrote some time back this is a times 1 minus epsilon square epsilon is the eccentricity and you have 1 plus epsilon cos of theta. This length is from here to here the place where theta is Pi by 2 is a times 1 minus epsilon square that is correct this is the closest distance and this is a times 1 minus epsilon and from the centre of the ellipse to the focus is a times epsilon.

Let me denote dead also buy these, perfect and this the rectangle which you see here is 2a from here to there and this length is 2b from here to there. Now, we want to know the magnitude of a so you can choose any point in the orbit and try to calculate the vector a, so you find out what is the momentum at that point let us say you choose the point over here. So, you should find out what is momentum there and angular momentum is anyway fix so there is nothing to get from the orbit and then you find out what is the r for that point.

So, this point and of course it will be best, if you choose this point to calculate a it does not matter where you calculate because it is going to be a constant. It is not going to change. So, let us look at this point and evaluate a in at this point. So, it is here so let maybe I can give expression on next page.

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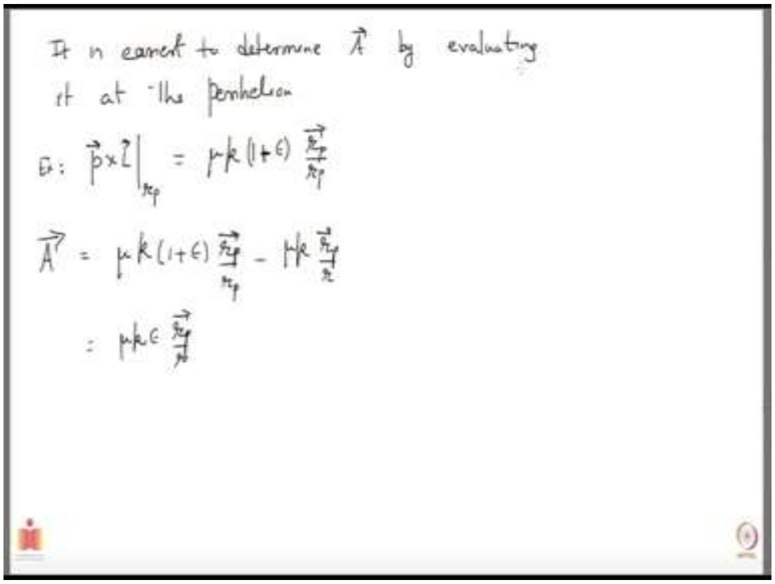


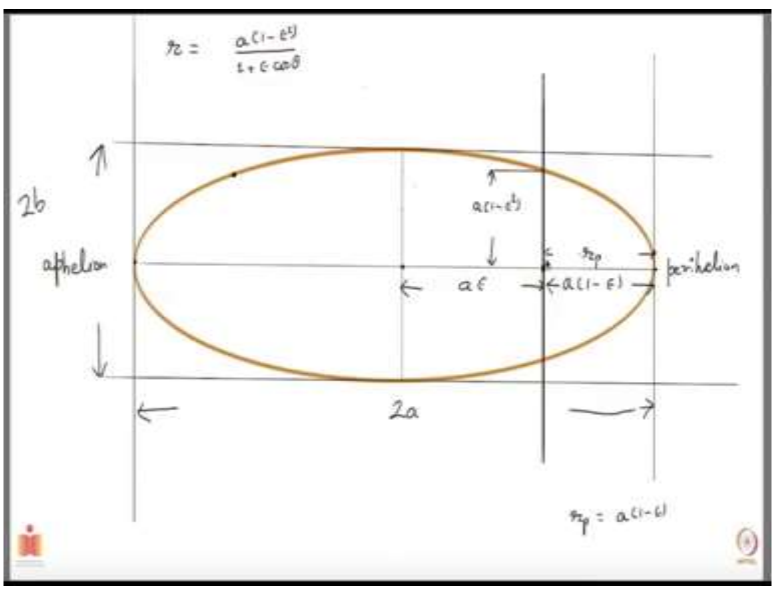

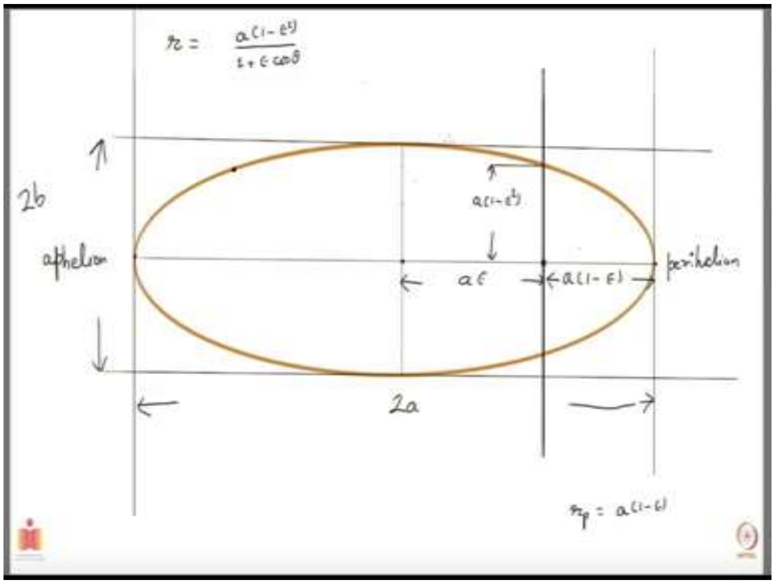


So, it is easiest to determine A by looking at by evaluating it at the perihelion. So, let us do that exercise number 1 trivial; almost trivial. P cross L, we want to evaluate this as at rp so by p I mean perihelion. So, this is the this length from focus to perihelion I am calling rp which is a1 minus epsilon. Please show that this is equal to the following mu k 1 plus epsilon a k is the thing which appears in the potential and (())(16:55) reduced mass epsilon is the eccentricity times rp over rp.

So that is what you should get and maybe I should emphasize that, rp is a1 minus epsilon this is rp. So, that is rp I just removed because I did not want to clutter this page and all you will need to really use is that p and L so the momentum and angular momentum at perihelion would be having 90 degree see here the particle would be going in this direction and angular momentum is anyway I mean sorry angular momentum is anyway perpendicular to r so and p correspondingly.

So that is not the point what I wanted to say was at this play at this point the momentum is also perpendicular to the radial direction. So, this vector and this momentum they will be having 90 degree for the same is not true at any other I mean for example here, or is it, anyhow at this point it is true probably for (())(18:53) may not be true I will I will think about it later.

So, that is one thing you please show and once you have done that our vector A is immediately available to us. So, A would be mu k times 1 plus epsilon rp over r that is the first term minus mu k rp over r and calculating that is also you know it is quite easy this not much and with that you will get the following.

So, you have mu k and 1 plus epsilon minus 1 will give you epsilon and then you have rp over r which is just the unit vector in the direction of perihelion. So, that is the Laplace Runge Lenz vector. So, I will stop this video here and will continue further in the next video.