Introduction to Classical Mechanics
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Lecture 35
Apsidal Distances, Eccentricity of Orbits

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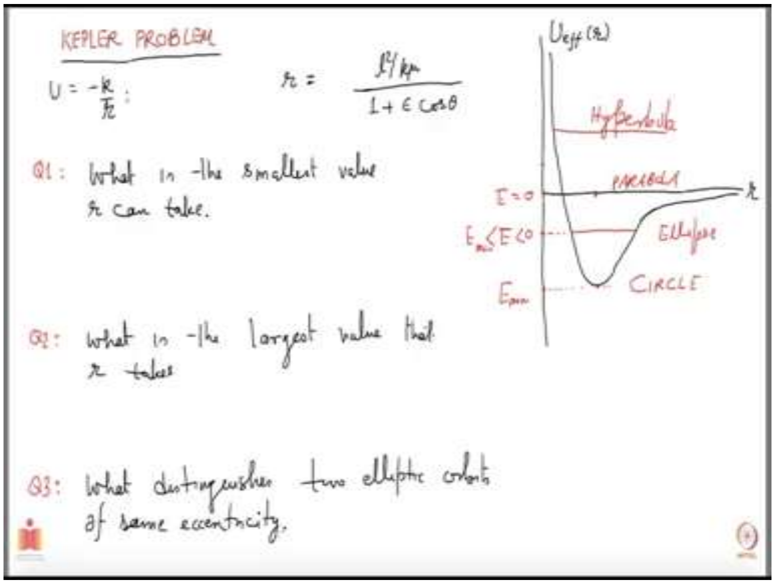

Let us summarize what we have done till now. So, we were looking at Kepler problem where the potential energy of the system is given by minus k over r and we had reduced the, I mean ofcourse we were doing two body problem and then we choose this potential and we found that the r is given by the following relation which are basically conic sections and you have here l square over k mu k and mu both are in the denominator of l square and then you have 1 over sorry 1 plus epsilon cos of theta.

So, that is what we had found I have here on the right hand side the graph of u effective versus r. Can we have already discussed that the solutions which we have found which is given here they provide the following orbits for the problem. So, if you are here, let me try to choose a different color. If you are at this value of energy that is a minimum then only one radius is possible which is this, and you have a circular orbit.

So let me your E minimum and you get a circle, circular orbit then if you are anywhere between this energy, minimum energy which is a negative value and 0, then you get an ellipse. If you are anywhere between this point and that point in energy then you get an ellipse. So if you are here, you get if energy value is this then you get an ellipse. If your E is equal to 0 let me write it here like this.

And recall if energy is 0, then you get a parabola. So, for this, you get a parabola. You see it comes here and then it returns back returns and goes towards infinity and never comes back. So this (leaves) leads to a parabola. And if your energy is higher than 0, if you are anywhere here or here or wherever you are that is also an open orbit, not just open it, it goes to infinity the particle returns to infinity and it is a hyperbola. So, that is what we had found.

Now let us think about what are the questions that we would like to ask given that we know this much. So I would encourage you to take a moment, maybe stop the video and think about things which you would like to ask further in this problem given that you have found that the orbits are parabolic, what are the questions that are coming to mind and let us list them.

So, you may stop it for a moment and return after you spend some time thinking about the questions you would like to ask. So, hopefully you have returned with a list of questions. And I will write down a small list which typically, we can ask right now. So, you may be interested in asking for example, question number 1 could be, yeah. So, your first question could be what is the smallest value of r that can be achieved?

Equivalently what is the closest what is the shortest distance between the particles m1 and m2 that you are going to see in the system, so that is question number 1. So let me write it down what is the smallest value r can take? So, that is a nice question to ask because you would, ofcourse like to know how close these 2 particles are going to come, right. Then you may ask what is the largest value that r can take?

How much what is the maximum separation between them that is going to be possible? What is the largest value that r takes? Now, you may also wish to think what distinguishes two elliptic orbits of same eccentricity. Ofcourse you will think of the scale one could be smaller another could be larger, they may have same eccentricity, but what is controlling them? So we will make this question a little bit more precise later.

But will for now, we will leave it a little vague. But nevertheless I will write it. What distinguishes two elliptic orbits of same eccentricity. We will make this question more precise after sometime. Maybe I can say a few words to the students who are listening to this video and if there are some faculty who are taking it for FDP, they may ignore whatever I am saying and excuse me, because I would like to address the younger laud, the students for a while.

So please, excuse me. There is one experiment that I sometimes do in once in a given course and that is the following. So, usually, after some, I have spent some weeks and I feel that students are getting along I would typically I mean I will choose one day on which I will decide that I will ask the students to leave aside their notebooks, and close your eyes, close their eyes and put their head on desk and try to think of questions that they would like to ask.

In the connection of whatever we are studying on that day and I will usually give them 5 to 7 minutes to think. So they are not allowed to write anything, they are not allowed to talk, they just have to think. And after that those 7 minutes, they will be allowed to write it on a piece of paper, which they will either read out in the class or they will hand it over to me and I have found that students find this interesting because usually, they do not get that much opportunity to ask to themselves the questions they know that the question they may be interested in at that moment about the subject they are learning.

So if you are a student who is least taking this course, I would definitely encourage that you sometimes just leave the book, leave the notes and think of things which you would like to ask. For example, here you might have come up with these questions, the simplest ones or even better ones. When I few minutes ago I asked you to think of some questions. Okay anyhow so let us proceed further. So, let us try to answer these questions which you have written down. Let us go to the next sheet here.

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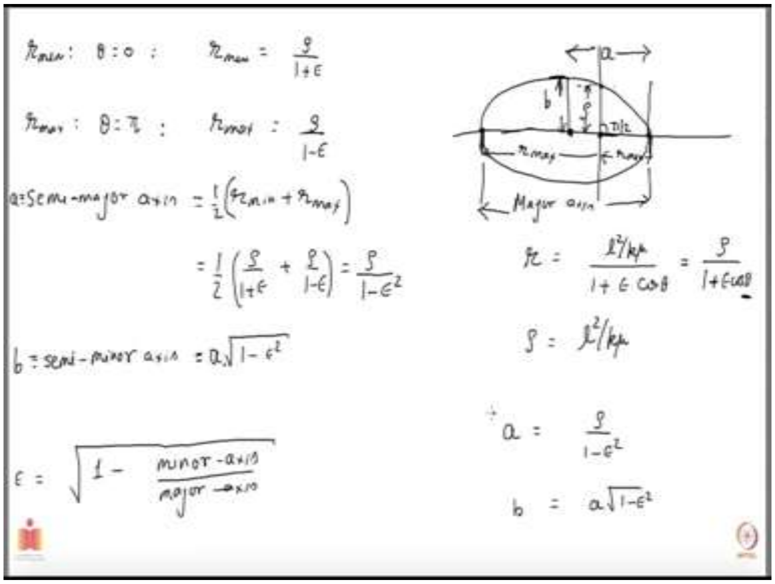

So, that is our orbit, the origin is here and my question is r is equal to l square over k mu 1 plus epsilon cos of theta and theta is measured with respect to this line. So let us say theta is Pi by 2, let us see what is that. When theta is Pi by 2 cos theta is 0 and you are left with 1 in the denominator and your r is l square or k mu which means this distance from here to here which I will call rho that is what you get for Pi by 2 which means rho is l square over k mu.

So, I will then rewrite this equation, this equation of orbit as rho over 1 plus epsilon cos of theta okay that is fine. Later, we will see that this is not still the best thing to write. But anyway, it tells you what the numerator is in terms of this figure. But this is still not the best way of writing the equation. Anyway, so let us ask about r minimum. Well, that r minimum occurs when theta is 0, because when theta is 0, it is going to be here.

And we have already seen that our curve is like this, it is not that way. So you are r minimum gives you r minimum you get by putting theta to be 0 and you put theta to be 0 here, it is rho over 1 plus epsilon. So, you get for theta equals to 0 and your r minimum is rho over 1 plus epsilon that is nice. Maximum value of r is taken when theta is Pi, so from here you go to there, this angle.

So, when theta is pi you are here and r maximum is when theta is Pi, cos theta is minus 1 and you get rho over 1 minus epsilon, okay so that is nice. So, here it is r minimum that is r maximum. So, r maximum is from here to there, this is r minimum from this point to the origin and clearly this entire distance from minimum to maximum is somewhere in the center of this ellipse, this is the center of the ellipse and the distance from the center to one end is what you call semi-major axis.

And similarly on that side you call semi-minor axis. So, this the full distance from here to there is r max plus r minimum let me not write it, it is cluttering. So, this is major axis this length and half of it would be semi-major axis. So, my semi major axis is how much? Is r max plus r minimum divided by 2 because then I count from here. So I have r minimum plus r maximum the whole thing divided by 2.

And what is that it is rho over 1 plus epsilon plus rho over 1 minus epsilon which is rho epsilon will okay 1 plus epsilon plus 1 minus epsilon will give you a 2 and there is a factor of half which I have missed. So you have rho over 2, but when they go up in the numerator, they bring a factor of 2 which will cancel the half here this two the denominator and you get rho over 1 minus epsilon square that is your same semi major axis.

Let us check whether it is correct; yes that is correct. Now, how can I find out the semi minor axis meaning this distance? Let me give semi major axis a name, I will call it A and let us call this one B semi minor axis. So B is defined to be semi minor axis and how can I find it out? Well, you can find it out by recalling what the definition of eccentricity is, do you recall that.

So if you recall your epsilon is defined by 1 minus minor axis or semi minor axis it does not matter because there is just a factor of 2 divided by major axis if you take semi minor and semi major the effect of 2 will cancel in the numerator and the denominator so it does not really matter that is the definition of eccentricity.

So I can use this epsilon to write that semi minor axis is 1 minus epsilon square in the square root times semi major axis which I am calling now as A so let me put the A here. So my B is A times 1 minus epsilon square, okay that is good and that is also good. So, let me just write it down neatly here again, my, let me write it down like this from here to there is A and this is B from here.

Now, before I leave, for easy reference, I will write here that A is rho over 1 minus epsilon square and B is A times 1 minus epsilon square. Let us leave it like this, I will give you two very small exercises to yeah, very simple.

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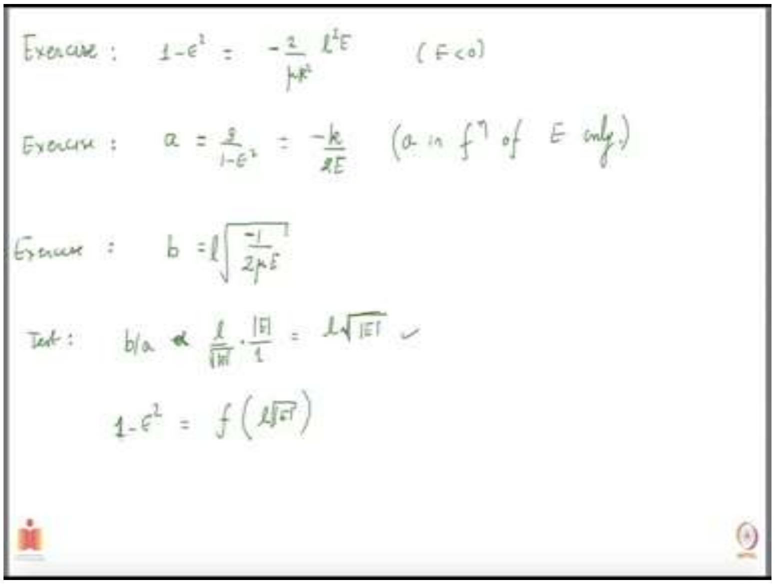

So exercise, please check that 1 minus epsilon square is minus, how should I put it? Yes, minus 2 over mu k square l square times E. Let us see whether at least the sign part is okay. Our epsilon is less than 1 because it is an ellipse, so epsilon square will also be less than 1, which means 1 minus epsilon square would be positive.

On the right hand side, even though you have negative sign, but remember, E is negative which makes the entire thing positive. So this looks fine as far as sign is concerned. Let me put epsilon is less than 0, because it is an ellipse. Second exercise; please check the semi major axis which we denote by A which is our rho over 1 minus epsilon square is minus k over 2E.

Again, E is negative, so minus k over 2 is a positive quantity which is good because we need A to be positive, it is a length. Also note that the length of semi major axis does not depend on l. Meaning when you fire the particle mu, the reduced particle with reduced mass mu the energy of it or the velocity of it is going to determine what the semi major axis would be what will be the length on along the theta equal to 0 direction from minus Pi to Pi minus Pi to 0, that length of semi major axis is completely determined by the energy you give.

And angular momentum has no role in determining that. So let me write it down. A; A is a function of E only I am not saying anything about K because that is given by the problem that is not determined by the initial conditions or that is given by the problem. Another exercise let us find out what the semi minor axis is in terms of the parameters which you control, which is the angular momentum and energy and it turns out to be the following.

It is minus 1 over 2 mu E and that sits in the square root and you have an l here, minus n is fine, because you have energy here which is which makes positive, that is nice. And let us do a quick test to see if these results which I am giving you are at some level okay. Let us do that test. So I am not going to worry about the factors of l and mu. But I will do some checks to see whether these are ok or not.

So let us look at let me call it a test. I am testing it. So let us look at b over a. Remember b over a is related to eccentricity, let us go back here. You remember epsilon is minor axis or major axis. So I am just want to calculate b over a from that is that is perfect. Yes, that is good. So let me cover b over a and let us see what it is. So b over a will be proportional to where is b, b is this so it brings l over square root of E.

So l over square root of, let me write mod of E, right now, instead of carrying of minus sign, and then I have to divide by a, and what is a; a goes as 1 over E, 1 over modulus of E. That is how it goes which is l times square root of modulus of E, that is what b and a are giving and if I have not made a calculation mistake, then as far as l and E dependence is concerned in here it should be the same as you should get in 1 minus epsilon square.

So let us see what our 1 minus epsilon square is. My 1 minus epsilon square is a function of l modulus of E in the square root, this thing square. But anyway, this is a function of this, so which is, which means that my b over a is coming out to be consistent with what I expected of the eccentricity. And so, this is these are the kind of small tests we should keep doing during calculations to be sure that we are not making some mistakes.

Okay that is fine, now let us ask the following. Let us go back to our third question what was or distinguishes two elliptic orbits of same eccentricity. So, what I am really trying to say ask is the following. Let us say I have fired my particle mu with some energy and some angular momentum and I get some orbit of some ellipticity epsilon.

Now, what I want to do is again repeat the same thing I want to fire my particle mu with a different energy and ofcourse, I should fire with different angular momentum such that I get the same eccentricity. So, I want to the same eccentricity as before but I want to have a different energy and then how should my angular momentum be related that is what I want to ask.

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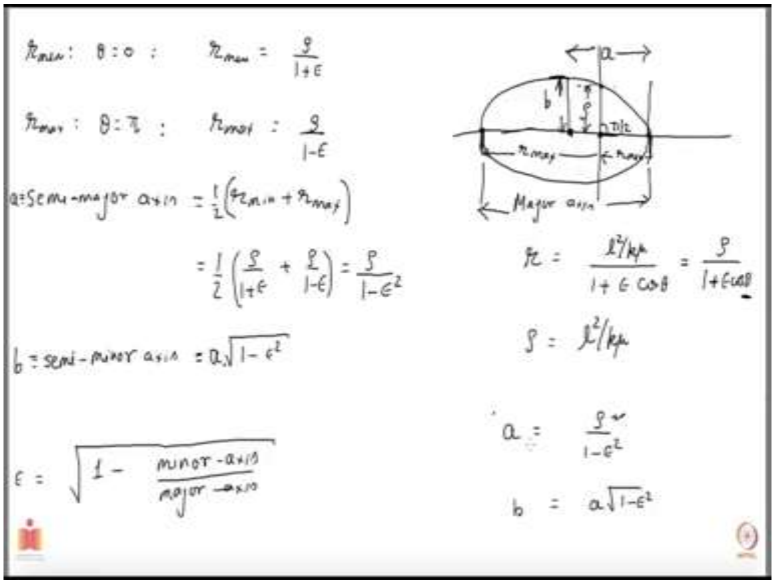

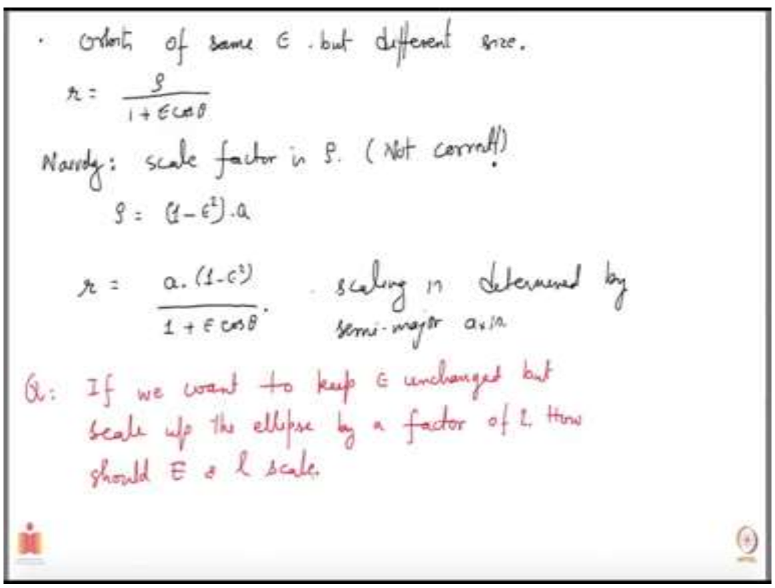

So, let me write or, so I want to write is, yes. So what I basically want is orbits of same eccentricity but different size this what basically I am asking, you see. If you draw an ellipse on a sheet of paper and you photocopy it and zoom it by 2 times, or 10 times or whatever the eccentricity is same for all these copies. The basic ellipse is still the same, what is different between all these copies is the scale.

And that is what I mean here that I keep the epsilon same and but the orbits are of different size or equivalently different scale. So very naively, I would think that because my r is given by the following my remember I wrote r as rho over 1 plus epsilon cos of theta, you might think that rho is the scale because epsilon remains unchanged and rho is the one which will determine the scale. And where rho is given by l square over k mu that is the naive expectation, if we are not being very careful.

So naively scale is controlled by or not just scaling is rho, skill factor is rho. But this is not correct but this is not correct, this naive expectation is obviously not correct. Because I should ensure that if there are any factors of a over b which are in rho, I should factor them out, otherwise, I will be putting in unnecessary see if you if I divide by this thing by epsilon then the scaling factor changes, so we have to be careful in what we are doing. So let us put it in the right form.

Let me write not correct and so I should recall that my rho is 1 minus epsilon square times the semi major axis. Let us go back and check whether we have encountered this before. Yes, indeed. So your rho here is 1 minus epsilon square times a, I should substitute that here. Now you see, you have I have pulled out of rho a factor which does not change when you scale up the ellipse because the epsilon is not going to change.

And that is what I was saying what is wrong in our naive expectation. So my rho, so my r is a times 1 minus epsilon square over 1 plus epsilon cos of theta. So, scaling is determined by the major axis, semi major axis let us say, is determined by semi major axis. Now let us ask what should we ask? Let us ask if you want to keep epsilon unchanged, but scale up the ellipse by a factor of 2.

How should you change your E and l after all they are all determined in terms of E and l, the energy and angle momentum is all you have in your hands. So, if I want to scale it by a factor of 2 what should you do? So here question if we want to keep epsilon unchanged but scale up the ellipse by a factor of 2, how should my E and l scale? How should E and l scale?

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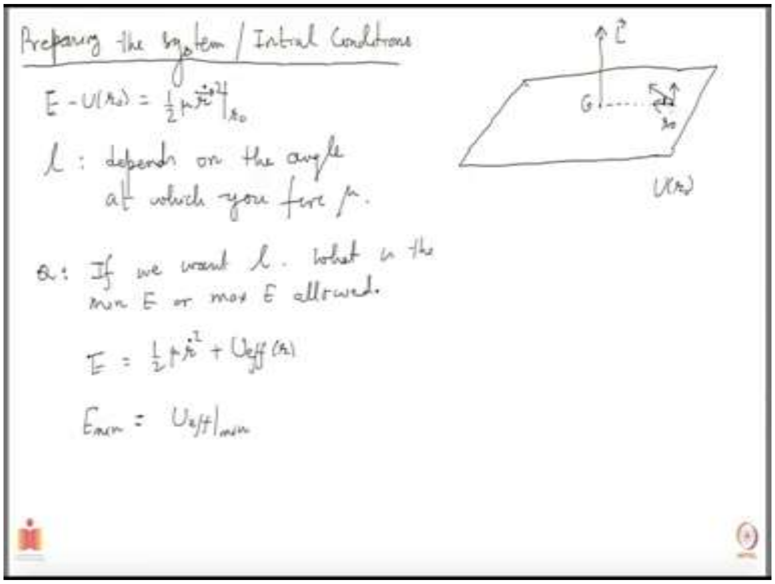

Let me go back and try to say what I really mean by this question. Where was it, somewhere here, here. Let us say you fired a particle from here some distance away with some energy E and you fired at some angle so that it has angular momentum l and you get an ellipse. You got an ellipse of some eccentricity epsilon, now take that ellipse and just photocopy it with a zoom of 2.

Now you get another ellipse with the scale factor 2 what we are asking is to get that nu ellipse. What should be the energy with which I should fire and what should be the angular momentum? Okay that is what we are asking.

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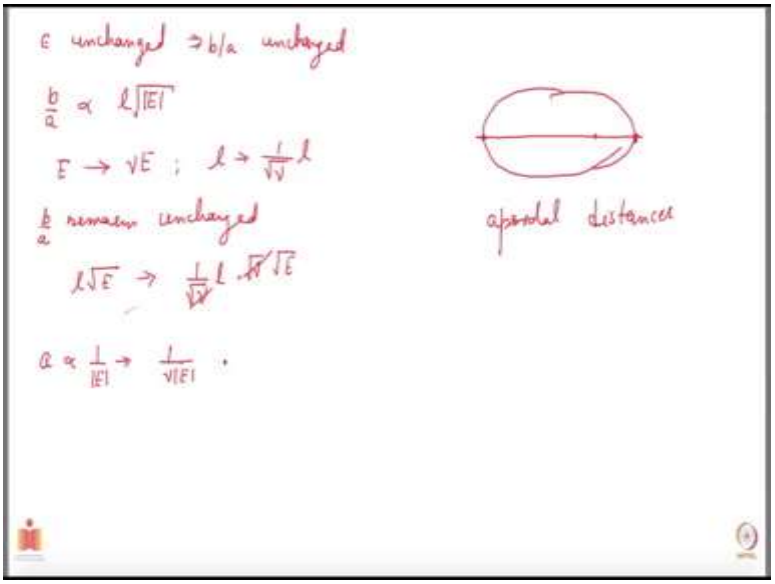

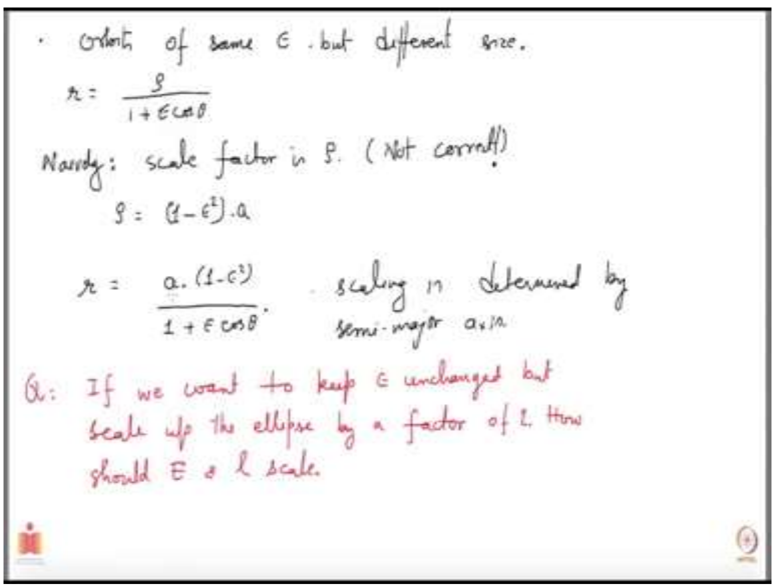

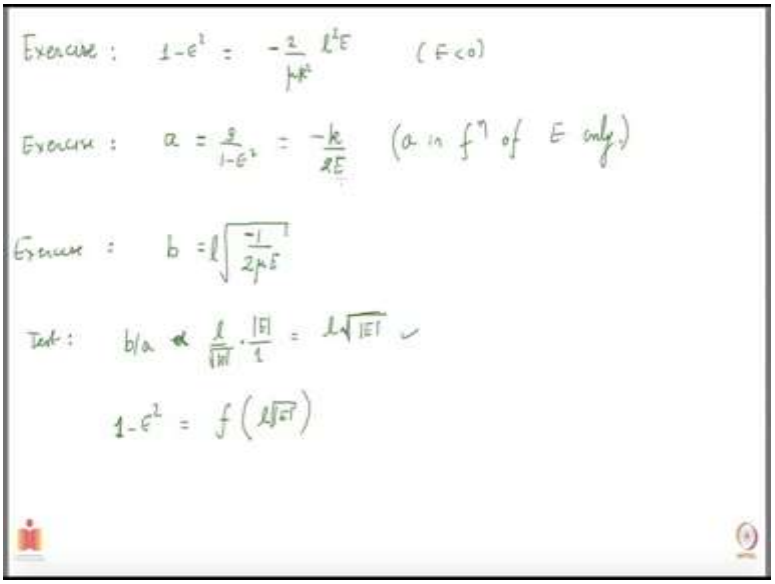

So, clearly because my epsilon is going to be unchanged, it means that the ratio of semi major axis and semi minor axis is going to be unchanged. So, epsilon unchanged implies my b over a to be unchanged. Now, b over a, you can quickly see that goes as l times square root of E, if you divide b over a that is what you get.

Now, let us say I take energy and scale it up by a factor of nu. So I make E to be nu times E. Then if I take angular momentum, and scale it as 1 over square root of nu, so I reduce it by a factor of 1 over square root square root of nu times original l. If I do this, then my b over a does not change. Unchanged, right, that is correct. So I will go as nu sorry, what I have done, yes, E goes as nu.

So you get a square root of E here, square root of nu times e here, let me write it anyway. So l times square root of E goes as l becomes 1 over nu l and E becomes square root of nu e, and these 2 cancel, so you get the same thing back. So that is correct, that is correct, very good. Now what? Yeah, that is correct, okay. Now, I wanted to ask what I would like to know now is given this situation where I have scaled the energies E and l by these factors, what is the nu semi major axis?

Because that is what is going to determine the scale of the ellipse, right? You remember this let us see, somewhere here. Yes, here you have a. So if I know how a has changed, I will know the answer. And let us go back and see what is a, a goes as 1 over E. So my a goes as or is proportional to 1 over modulus of E. That is what we have saw just now, to be sure, let us check once again, yes correct; a goes as 1 over mod of E which means under the scaling I have done my this will become nu times.

So, it will get scaled by a factor of 1 over nu, that is looking fine to me the size of 1 by E so it goes by a factor of nu. I think I wrote wrongly in my note, so this is correct here. So, that is how it happens. And let me also throw in a bit of nomenclature, these r minimum and r maximum they are called our apsidal distances. Let me draw this. So, this point and that point, they are called apsidal distances. We will continue and talk more about center force problem in the next lecture, so see you then next.