Introduction to Classical Mechanics

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Lecture 34

Orbits in Kepler Problem

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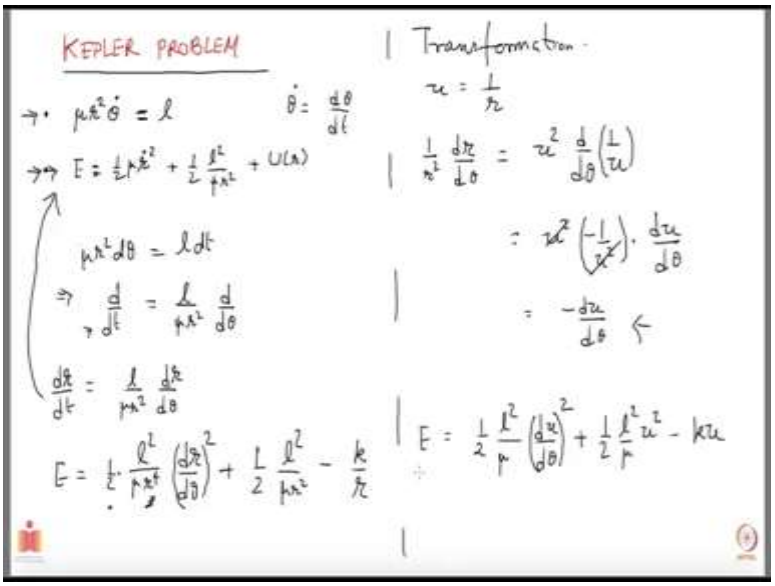

Okay let us return to the Kepler Problem which is basically choosing potential which falls off as 1 over R. For example, gravitational potential, so we had seen that we could reduce two body problem to a problem involving one mass mu, particle of mass mu which is moving in a central force field and then using angular momentum conservation we could reduce the problem to motion in a plane that is what we saw.

And we had also found that the angular momentum provides us with this relation. So, l is a constant which is determined by the initial conditions or which is basically the angular momentum of your particle mu and theta dot is theta d theta over dt. We also had the first integral which is the conservation of energy which is E is equal to half mu r dot square plus half l square over mu r square plus the potential energy which falls off as 1 over R in the case of Kepler problem.

Now, in this lecture I am interested in looking at the orbit of this particle mu. So, I am not so much interested right now in finding at what time where it is rather I am interested in finding out the shape of the, the trajectory, that is what I want to find out which is basically in the plane.

What is given a theta what is r apparently given r what is theta, so that is what we would like to see.

So, for that what I will do is I will take this first equation and from there I can write down I can convert this theta dot is d theta over dt, right. So, I have mu r square d theta is equal to l dt, so I can convert changes in time as changes in theta. So I can write this into the following form; d over dt l over mu r square d over d theta. So, I can take convert the derivative with respect to time into a derivative with respect to theta.

So, with this now I want to manipulate this conservation of energy equation and turn r dot into a derivative with respect to theta. So I will use this 1 and write dr over dt as l over mu r square d over d theta and r is here, okay that is nice. Now with this my first integral this one becomes the following. So I substitute it in here and get the following; E is equal to half we have r dot square so I should square this thing, there is mu here and there is a mu here which will be square so I will get 1 over mu and that is what I write here.

So mu then r square become r to the 4, l will become l square and you will have dr over d theta whole square, this entire thing will be square plus, let us look at the second term, it remains as it is half l square over mu r square. And our potential we have taken to be minus k over r. Now I am going to do a nice transformation on this.

It is a very nice transformation, because you will see all these powers of r to the 4 which are multiplying these derivatives they will disappear. So what we do is we take u to be 1 over r that is out new variable. Now, let us look at what happens at, what happens to dr over d theta. So, let us look at 1 over. So you have r to the 4 here, dr over d theta square. So let us look at only 1 over r square dr over d theta.

So I want to look at 1 over r square dr over d theta and let us find out what this is. So if I substitute u here in this expression, 1 over r square becomes u square and you have d over d theta of 1 over u. Now, if I do the derivative, I will get mu square is there anyway, it will give me minus 1 over u square times du over d theta. And these u squares cancel and leave behind minus du over d theta that is very nice.

Because now my, this entire thing will not have any u square multiplying the derivative pieces, so if I substitute here, in here, this quantity, I get E is equal to half l square over mu that is correct and my r 4 is gone, the minus sign goes away because of the squaring and I am left with du over d theta squared plus half l square over mu 1 over r square becomes u square that is correct minus k u. It looks much simpler compared to the previous equation, that is nice.

Now, energy is a constant of motion. So, whichever way your particle is moving around in the field wherever it is, at any time, any point of time is going to remain constant. Meaning if I take a derivative of E with respect to theta, the left hand side the E will going, is going to be killed and it will give me 0. So let us, I think I will have to write it again on the next sheet. Let me do it quickly.

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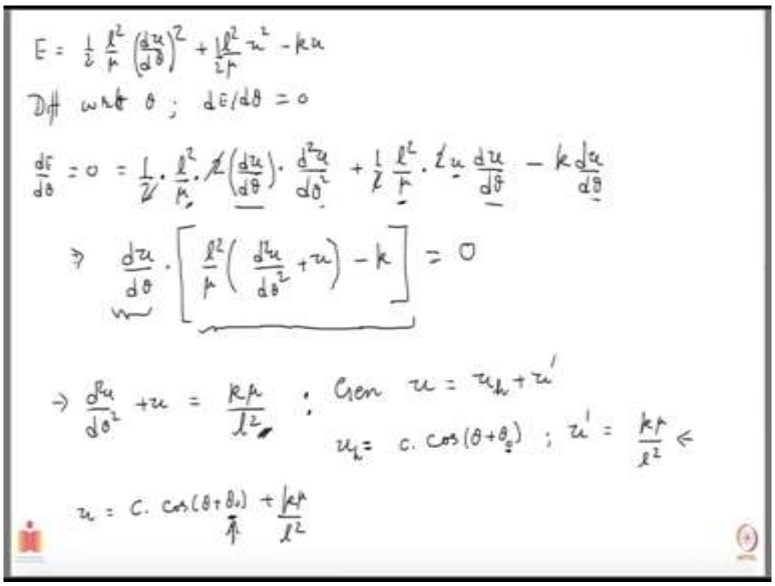

So, I want to take a derivative of this previous expression with respect to theta. Let me write it because needed here half l square over mu du over d theta squared plus l square over again half mu and I have a u square minus ku. So differentiate with respect to theta and use dE over d theta to be 0. Now if I do so I get dE over d theta is equal to 0. This is equal to so I have a half when I take a derivative a 2 will be pulled out which will cancel the half let me write it anyway.

So, you get 2 du over d theta times second derivative of this right d2u over d theta square. Then you have half l square over mu. And then you get 2 u du over d theta minus k du over d theta that is nice, each of them has a piece du over d theta which I can pull out let me cancel this half, let me cancel this factor of half also and yeah. So, this implies that I can write the above equation in the following manner.

Let me see, correct, all is good till now. So I have l square over mu this term here, this is the same thing in here and I have d2u over d theta square I have taken this now I should take this piece here plus u. I have already taken du over d theta outside. Let me put a square bracket and then I have minus k, correct? That is correct. So, you have 2 possibilities; either this piece equal is equal to 0, or this piece is equal to 0.

Now, if du over d theta is 0 it just means that u is the constant, constant which means r is a constant, which means a circle, so a circular orbit which is indeed a solution. Let us look at this piece then which has more interesting things to offer. Let us look at this. Let me write it in the following form. So I get d2u over d theta square plus u is equal to k mu over l square that is nice, our equation for u is basically yeah remember, u is 1 over r.

So you have a equation of your orbit now in front of you okay it is there. Now, if the right hand side was absent k mu over l square this in homogeneous parts part was absent, then the homogeneous equation is very familiar to you that is, just the equation of oscillator, simple harmonic motion. So, it will have the general solution will be u is equal to u of the homogeneous part plus the particular solution because of the inhomogeneous part on the right hand side.

And your u homogeneous is easy, you already know that, u will be some constant, which I want to call, let me call it, let me call it C. C; C does not look good okay I should call it yeah, let me call it C, C times cos of theta plus some theta naught let us see here. See, if this was d2u over dt square plus omega square u, then the solution would have been some constant times cos of omega t, right?

Now instead of t you have theta so that is why you have omega theta, but omega is 1 because this coefficient is 1, so that is why you get this and then ofcourse, you have an arbitrary phase part, the equivalent of that. And then you have the amplitude part here. So that is your u homogeneous. And what is your u prime? U prime the particular solution, so it is clear that u prime is just k mu over l square. If you take this, this thing and put in your equation this d2u over d theta square will immediately vanish because u prime is just a constant and then clearly your u prime is just k mu over l square. So, that is correct. So our general solution u is c some constant plus k mu over l square.

Now, can this constant c be arbitrary? Looks like it can be because you have solved this second order differential equation, but look at your original problem of that one particle moving in center field and you have already used energy and angular momentum to fix certain constants. So, should we expect c to be arbitrarily an arbitrary constant or it will get determined by the energy and angular momentum in the problem.

So, I will encourage you to think about it and convince yourself that c cannot be arbitrary that is 1 thing but anyway I am going to explicitly determine it. And the second thing would be about the theta naught. I will come back to this in more, I mean I will give an explanation of theta naught later towards after we have found a trajectory, the orbit, okay we will come back to that. So, that is nice, which means let me just write down this expression in terms of r now, because that is what the real thing is.

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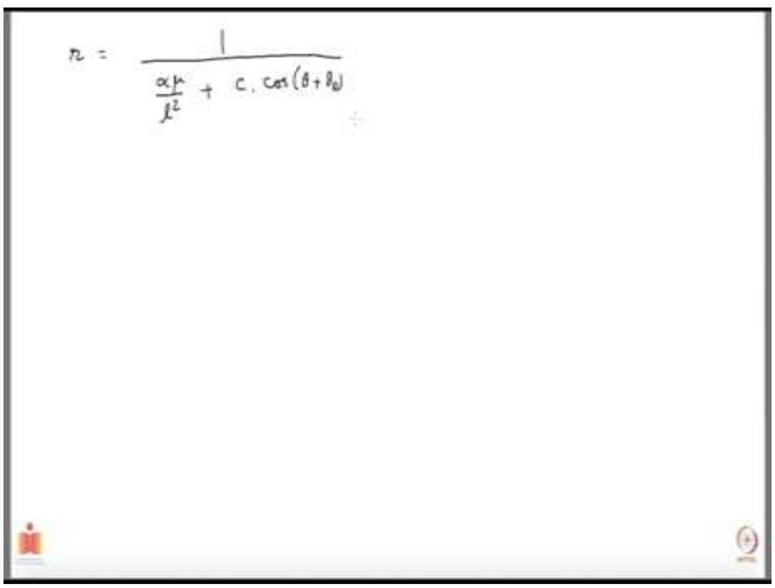

So, writing the previous expression in terms of r I get r is equal to 1 over alpha mu over l square plus c cos of theta plus theta 0 and this is very familiar to you now, because depending on what the value c takes and all these things, this will be either a parabola, a circle or an ellipse or a hyperbola that is what you have seen.

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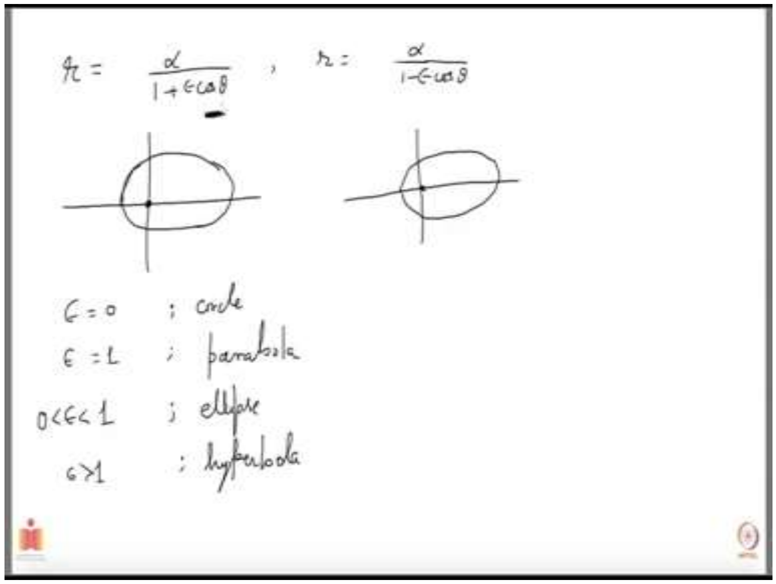

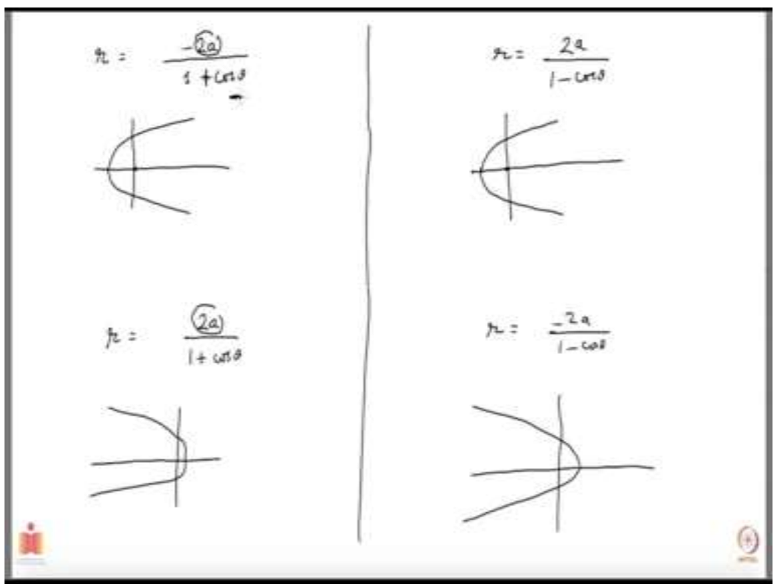

Let me go back few slides and show you this is, you see here, this was what we had for an ellipse. And that is what we had for parabola. And ofcourse circle is part of it if epsilon is 0. So, that is what we have found.

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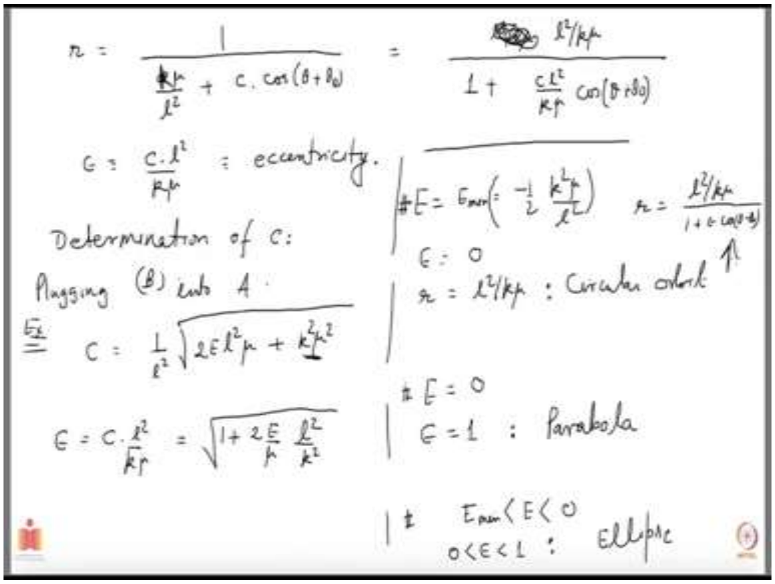

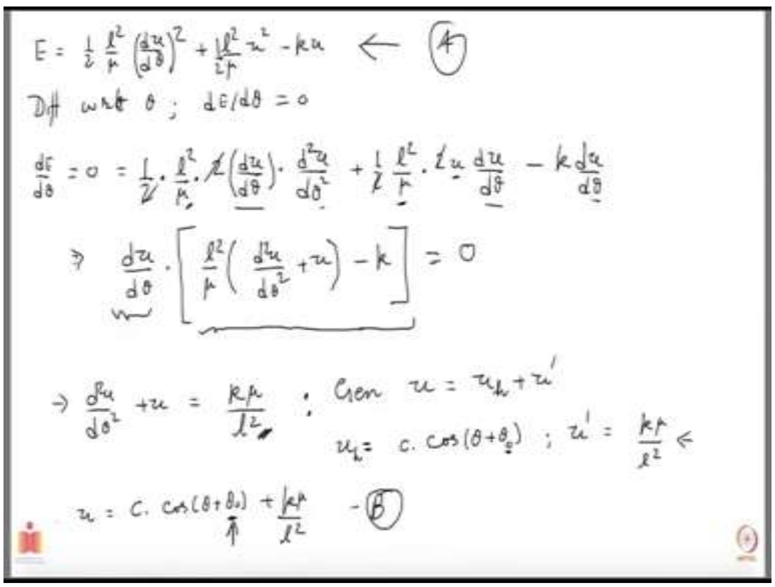

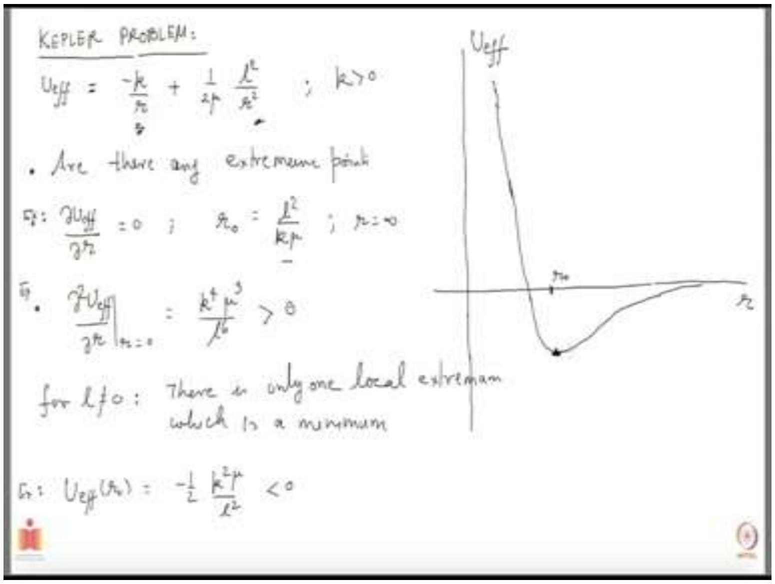

We have found orbits to be conic sections. These are all the polar equations of conic sections. Let me do a little bit of algebra and write it as. So, I take out l from your l square as common and put it in the numerator which becomes l square over l alpha mu. What is alpha? Okay alpha, alpha I see have changed compared to my notes I have changed k alpha to k here.

So, I should use k, k here also let me, what happened here so this should be k, k mu over l square and this should become l square over k mu and this will be 1 plus c l square over k mu, correct? That is correct cos of theta plus theta naught, okay that is good. Now, because you know what these things are, let me define epsilon which we defined as the eccentricity earlier when we were looking at the equations of conic sections in polar coordinates.

So, the same thing I do I define c l square over k mu that is what is the eccentricity. Looks something is wrong, this is okay why I think it is not correct. Let me proceed I think it is what I have written is correct but it does not match with my something in my note. Anyway let us go ahead let us see. Now I should determine, now let me determine the constant C.

Now that is not difficult because all I have to do is take my r or u whichever you like u will be easier take this expression and plug it in here let me write it, let me call this equation A this as equation B. So, I want to plug a B in A and when I take all these derivatives and put u there I will determine E. So, from plugging B into A I will get the following. Let me tell you directly the answer.

You will get c to be 1 over l square then you have that is what you should check, 2E l square mu plus k square mu square. Please do this exercise that is nice, correct. Yes I realized there something small mistake instead of multiplication I have written division somewhere in my note, all is good here what I am doing this is my c, correct. So you see c is this constant c is not arbitrary. It is completely determined in terms of your angular momentum energy and the mass of the particle and k.

They have no freedom in choosing it, as you can in case of a simple harmonic oscillator, but here it is completely constrained. So what is my epsilon now, eccentricity? My epsilon is the following. So I take this c and I have l square over k mu and you can easily check that this is nothing but I think I have a better expression somewhere.

Yes, you can you can write this as 1 plus 2E over mu l square over k square that is what it will turn out to be we can check clearly. So, you can pull out k square mu square outside which will become k mu and the denominator and this is being divided by k mu, so that k mu goes away and then you have 1 over l square and there is a l square in the numerator so that also gets cancelled.

And what you are left with here is 1 plus 2E l square mu over k square mu square which is just this space. So, that is what your epsilon is that is the eccentricity that is good. Now let us look at our orbits. Now you remember the minimum energy that this system can take, let us go back I think we can find somewhere, E minimum is your u, effective minimum.

And I think we found u effective minimum also which is here, so minus half k square mu over l square let us see k square mu over l square and there is a minus half that is right. So, we had found that E minimum is minus half k square mu over l square. I hope that is correct, k square mu over l square that is correct. So, let us see what happens when E is a minimum, the minimum energy which is allowed.

If you put E equal to E minimum here then the epsilon becomes then your epsilon is how much? Let us see, you have minus half which multiplies the 2 and gives you minus 1. Then you have a k square which cancels this k square in the denominator, then you have a mu here in the numerator, it cancels this mu. And then you have l square in the denominator that cancels this l square in the numerator.

So you are left with a minus 1 and minus 1 plus 1 will give you 0 so your epsilon is going to be 0, and if you put in r is equal to some constant here, which is for u, where is it? This space, you have l square over k mu over 1 plus epsilon cos of theta minus theta naught. So, if epsilon is 0 your r is independent of theta it is a constant. Meaning you have a circular orbit which is consistent with what you already knew.

You knew that if you are at u minimum effective minimum or energy is equal to u effective minimum at this place then there is only one value far which is allowed. So your particle mu is moving in a circular orbit which is what you have got here which is good. It confirms that every the algebra we have done is correct. Now, let us look at next possibility E is equal to 0. So we want energy to be 0. Let us go back and check what we are talking about in this energy diagram.

So now I am here I am saying that energy is 0 and you see the, we already talked about this that the orbit is not closed. The particle you fire, it comes approaches smaller approaches the center. And then it starts returning back at the turning point. And it just goes away and never returns back. So the orbit is open and let us see what we get here for energy equal to 0.

So if I put E equal to 0, I get epsilon equal to 1, so I get epsilon equal to 1. And we already know what that orbit is. It is a parabola and parabolas are not close, they are open orbits okay that is nice. Now let us look at energies which are between E minimum and 0 and remember E minimum was negative. So I am looking at such energies. Now in that case, our epsilon or the eccentricity is between 0 and 1.

You can check, when epsilon is 0, it becomes 1. Otherwise, ofcourse, because E is negative, 1 always smaller than 1, but 1 more than 0 because I am not touching the point E minimum and for this you know that 1 an ellipse. This equation is of an ellipse.

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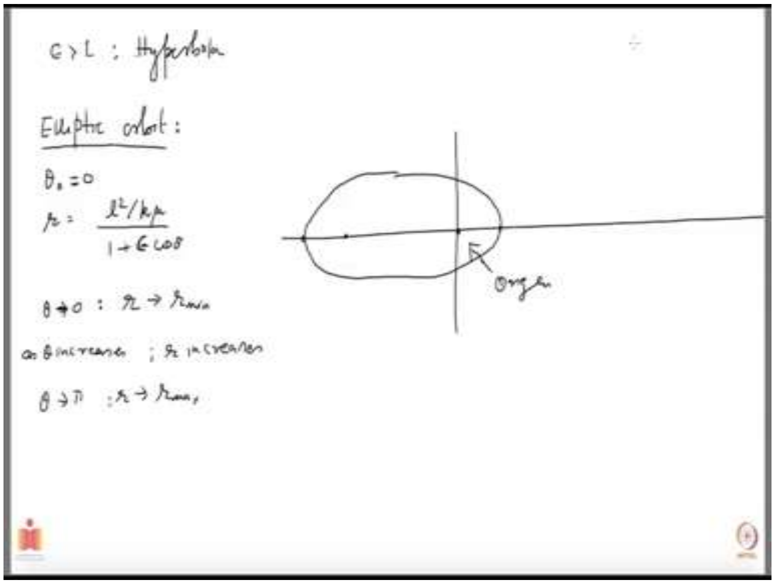

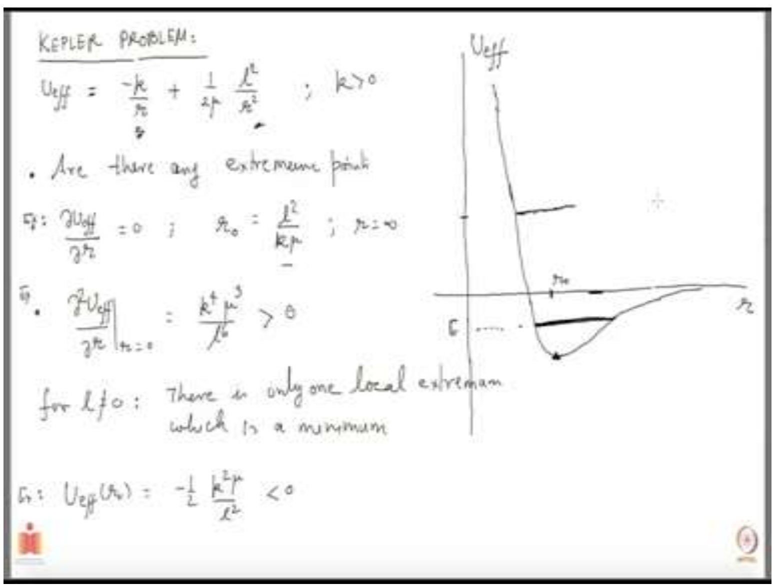

And if epsilon is greater than 1, you have a hyperbola. So, for ellipse, ellipse is a closed curve and hyperbola is open. And this is consistent with what we already expected from our energy diagram. So, if your energy is between 0 sorry minimum value allowed and 0, then your particle will be depending on what energy your system has will be in this region, for example, if I take the value of energy to be somewhere here and it is bounded, clearly it is bounded between these 2 points. But now, you know that the shape of the orbit is elliptical.

This one we have already talked parabola if you are here, and now, we also found that if my energy is more than 0, let us see the values somewhere here or any value above 0 then the particle will come. It will come towards the center, it will reach its minimum are allowed and then it will return but it will never come back again. It will be just lost, it will go to infinity. And that is and the shape of that orbit would be a hyperbola that is what our solution says.

So, this is good. Now, let me say a little more. Yes, so let us see where is my focus located in this? So, let us look at the elliptic orbit. See all I want to do right now is to tell you whether the ellipse is towards the left or towards the right, and that is what I want to say. I will make it more precisely what I am saying. So first, for a moment, I will put theta naught to be 0 I will come back to theta naught business later in a moment.

And if I do so, then my r is l square over (alpha) not alpha k mu over 1 plus epsilon cos of theta that is good. So it says if theta is equal to 0, then this cos theta cos 0 is 1. So it takes its maximum value which means the denominator takes its maximum value, which means r takes its minimum value. So at theta equal to 0, r goes to r minimum when theta goes to 0.

And as theta increases, your r increases, right? Because when theta is increasing, this thing is going down. So, r is going up that is what is happening. And when your theta is Pi, this will be minus 1 and this would take its minimum value allowed, and r will take its maximum value. So as theta goes to Pi, r goes to r maximum and then after that it starts again decreasing. So clearly, the situation is this.

So, you have for theta equal to 0 you should have r equal to r minimum, so let us say here somewhere and then as theta equal to Pi, you should have r equal to r maximum. So something of this sort. So the origin is here origin is not here. That is what I wanted to say and this is the place where your origin is, that is the focus point. So, that is all for now and we will continue later in the next class.