Introduction to Classical Mechanics
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Lecture – 32
Two-body problem, Conic Sections in Polar Coordinates

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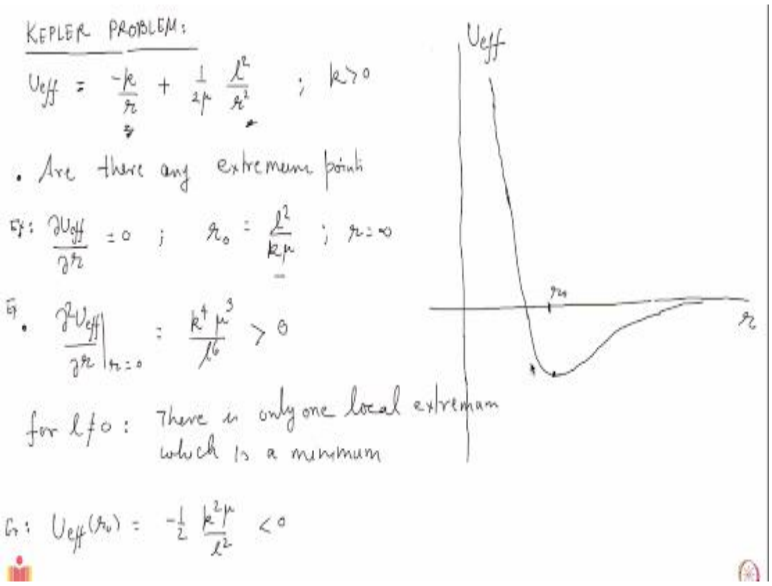

We were looking at the Kepler problem last time and we looked at the curve of effective potential and it looked like what you have on the right hand side here and we saw that it has a minimum which is it has only one minimum and there are no other extremum and the curve has the shape and we want to analyze this capital problem further, but before we do so let us just have some fun today with learning about certain some some curves in polar form.

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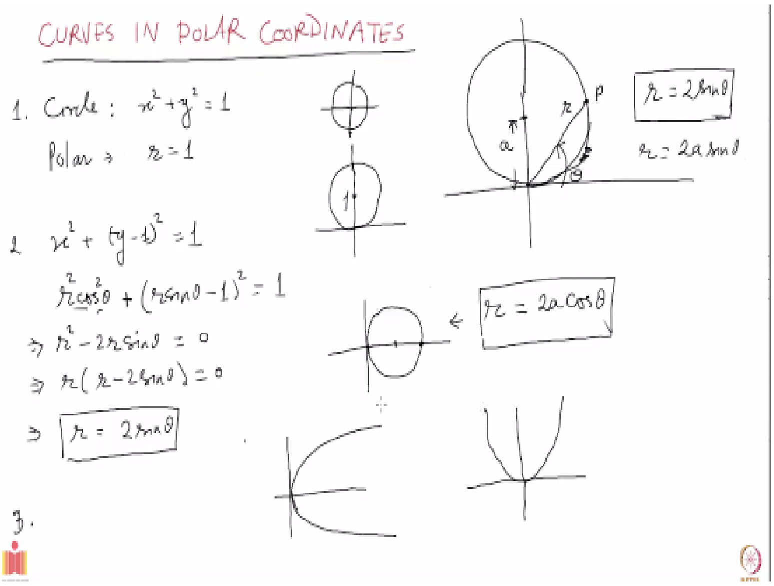

So, usually we are more familiar with curves in polar, in Cartesian coordinates and today I would like to discuss about them in polar coordinates. So, one of the simplest curves one of the simplest one is a circle and you know it is equation x square plus y square it is equal to 1 if it is a unit circle and it is centered at the origin. What is it look like in polar coordinate the same equation, the same circle? In polar it looks like r is equal to 1.

Now I will shift this circle along the y axis meaning I will take its center to lie at y equal to 1 so I shift it so that you have this. So the center is shifted to 1. Now what is the equation of this circle? So let us look at what it looks like in Cartesian it looks like x square plus y minus 1 square equal to 1. Let me draw the circle again so that those of you who are seeing it on smaller devices can see it easily.

And this is your center of the circle and this is the origin so circle is touching the origin and we are measuring with respect to here. So let us say this is some point p on the circle which has a radius r and makes an angle theta forgot this it makes an angle theta and I want to know the equation of the circle in r and theta. So, you put x equal to r cos theta, r cos theta and you square it then you put y to be r sin theta. And then you have minus 1 you square it and that should be equal to 1.

Now let us just do this very trivial algebra I am doing it because it is so trivial to show. So you open it up it gives you an r square sin square theta that I will combine with this term first term then you get a 1 and then the product, the cross product. So, I get this plus the first term from here gives me r square.

I have U sin square theta plus cos square theta equal to 1 then I have minus 2 r sin theta minus 2 r sin theta and then you get a 1 from here, but that cancels with that 1 here so I get a 0 on the right hand side which I can factor into r, r minus 2 sin of theta that is equal to 0 and which immediately gives you the equation to be r is equal to 2 sin theta. So, the equation of this is r is equal to 2 sin theta.

You can change the radius to something else let us say if the radius was not 1, but was a, then this will become r is equal to 2 a sin theta that is what it looks like in polar coordinates. So, if you are given sin theta and you plot it in the Cartesian coordinates you get the sinusoidal curve, but if you plot sin theta in polar coordinates you get a circle. Now we will ask before I let us ask what would the equation of the circle would be if you shifted the circle along the x axis, along the x axis if instead of this you had let us say this thing.

So, if you have a unit circle which is just touching the coordinate axis what will be the equation for this and you should find out that this has a polar equation r equal to 2 a cos theta let us see whether it looks reasonable. So when theta is 0, I should get r to be here which means 2a cos 0 is 1 so 2a looks good. Please check that this is what you get for this circle. Now next we would like to ask, what are the polar equations of parabola?

So, I would like to ask now if I am given a parabola what does it look like in polar coordinates this or this? So, let us look at this now.

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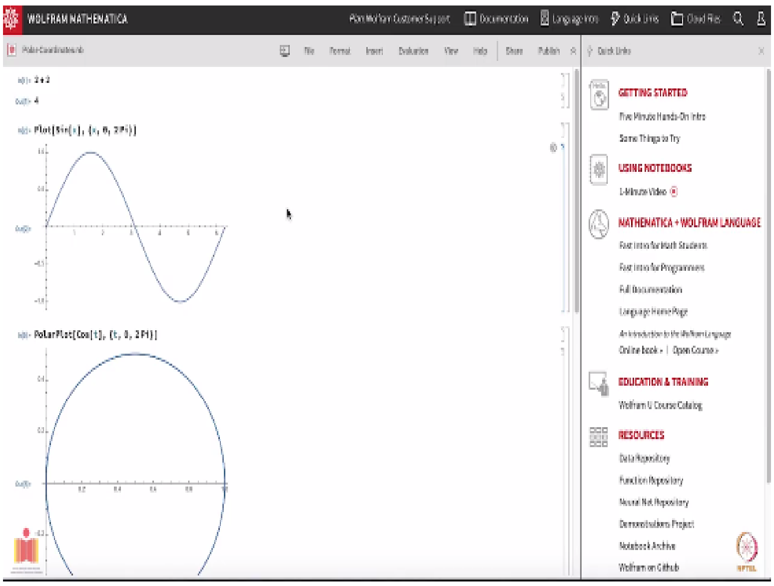

Before I move on to polar, polar equations of parabola, I want to encourage you guys to use some programming language to do for example plotting and solving things. So, this scale you should have and I have here Mathematica in front of me. I believe you can access Mathematica for free online. So there is some free freely available thing available from Walfram I checked the website so I thought that maybe I will show you how small things you can do here.

This one is not free this one is through my institute subscription, but you can search online. So, let us say I want to just let me show you addition of two numbers here. So, I write 2 plus 2 and I hit shift enter so I should be pressing shift and then hitting enter it calculates and you get 4 so that is the way it shows I hope you can see here easily let me see if I can anyhow. So here it is lying in 1 meaning I have input 1 here 2 plus 2 and output 1 is 4.

If you want to plot something let us say you have some function sin of x. So here is the syntax the sin should be written with capital S and the function argument has to be in square bracket. So that is what I am doing x ranges from let us say 0 to 2 Pi so capital P and for the plot so I am plotting this you can learn the syntax on your own I do not want to spend time on that I just want to encourage you to do this that is why I am doing it here.

So, you get the sin function. That is a plot of sin function which you know. Now, let us do our polar plot so, I I said that if you want to look at the circle it will look like 2a sin theta or 2 sin theta for a unit circle. So, I want to plot it and it is called polar plot if you want to plot in polar coordinates so Polar capital Plot so I close open and close the square brackets and what do I want to plot?

I want to plot sin of theta let me call theta as t and the values of t should run from 0 to 2 Pi and let us see what I get when I do this and you know what I should expect I should expect a circle. Remember sin theta is a circle in polar coordinates and that is what you get. When you see here it is touching the x axis on the circle is touching the x axis. You may also see let us say I change sin of t to cos of t.

I mean I can plot both the things together, but I do not want to do it right now so cos of t here let us see what do you expect? We expect the circle to be having its center on the x axis so this is what you get let us say nothing happens hence there has to be something mistakes because I have written cos t here it should have been cos. Let me again shift enter let us say perfect.

So, please one programming language which you can use to do all these analytical manipulations and Mathematica is one of them you can use anything, any programming language. I am not going to base any of the questions on Mathematica so if you do not learn this no problem at all, but if you learn it, it will be very good for you and given that it is free as far as my understanding, goes it will be very good for you.

So, we will continue with with parabola and also I will show you some plots in Mathematica for parabola after we have derived the equations.

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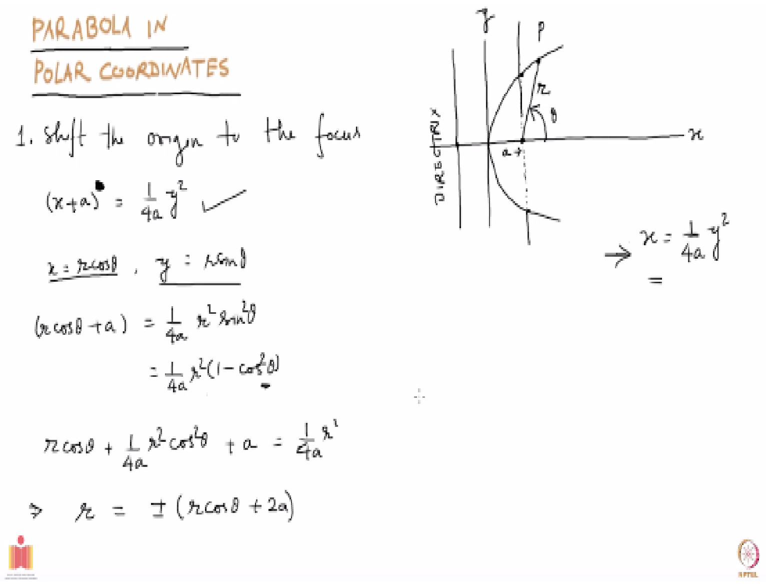

Let us recall what parabola is? It is a set of points which are at equal distance from a given point which we call focus and a given line which is called directrix. So, let us say this is my y axis, that is my x axis and I choose that given point to be or the focus to be here a distance a away from the origin and I want my parabola to have a have its vertex here at the origin. So, which means that if this point has to be equidistant from this point and the given line, then the directrix should also be a distance a away from the origin.

So, my directrix will be passing through this. So this is my directrix. So, let us say, I I draw a line parallel to the directrix or x axis from here, this length is 2a from here to here so I mark a length 2a here on this side and similarly 2a on that side. So, clearly because this length is equal to this length my parabola will pass through these two points as well and then I can draw my parabola to be like this and here.

So that is our parabola and you know what the equation it has in Cartesian coordinates it is x equal to 1 by 4 a y square. This is y. This is x that is fine all good. Now, let us find out what the equation looks like in polar coordinates, but before I do that I want to shift the origin from here to the focus. So, first I do shift the origin to the focal point to the focus so which means I start with this equation which I have here x is equal to 1 over 4a y square and shift my x axis such that focus is, the origin is here.

So, I should write x plus a whole square 1 upon 4a y square. So, after shifting the origin to the focus, I mistake here I get x plus a as 1 over 4a y square so here in this in this equation I have just shifted the origin. So now substitute x equal to r cos theta and y equal to r sin theta in this equation and I got I get r cos theta plus a is equal to 1 over 4a r square sin square theta which I can write as 1 over 4a r square 1 minus cos square theta.

So, I take this term cos square theta on the left hand side and get r cos of theta plus 1 over 4a r square cos square theta plus a equal to 1 over 4a r square and you will be able to solve it quadric equation and you can solve it for just complete the square and then you can write the following. So, from this you will be able to show that r is equal to plus or minus of r cos of theta plus 2a gets a trivial algebra that you have to do.

And of course you get two possible solutions from here which are the following maybe I should go to next page.

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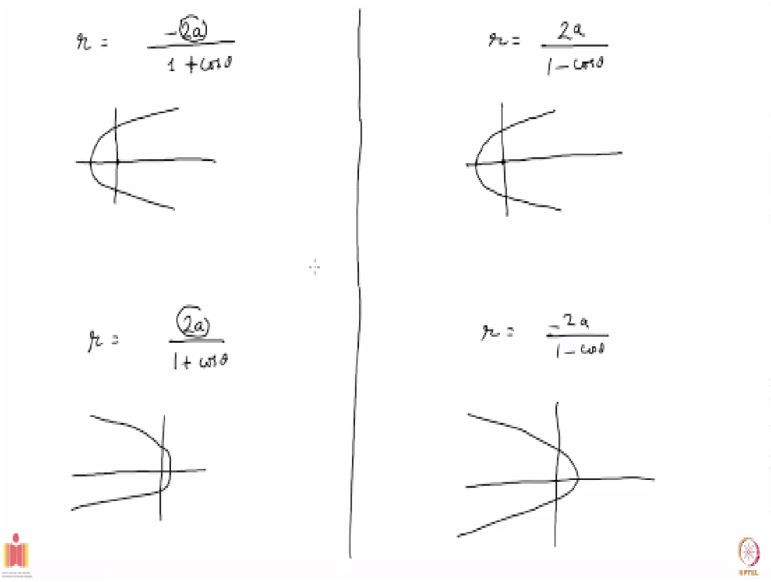

So you get one solution as r is equal to minus 2a remember a is positive 1 plus cos of theta that is one possibility another possibility is r is equal to 2a over 1 minus cos of theta and you should try to plot these two curves and you will get the following you already know what you will get because that is what you have constructed. So, this one will correspond to this which is what we started with this is the origin.

And this one also corresponds to the same thing no surprise because that is what you started with. So there are two, these two equations give the same parabola at the origin. I will encourage you to also check the following the different state of the above you choose r equal to 2a over 1 plus cos theta then you get the following. You get a parabola like this, but this time it is opening up on the left hand side.

And similarly here if you put r equal to minus 2a over 1 minus cos of theta then you get this parabola. So, that is the equation of parabola. So, you have some number in the numerator which controls the you see what this 2a is, look at the 2a here or here. This 2a it is just the distance of the vertex from the directrix that is what it is and then you have 1 plus cos theta or 1 minus cos theta in the denominator. Next I would look at an ellipse and write down its equation in polar coordinates.