Introduction to Classical Mechanics
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Lecture 31
Two-body problem, Kepler problem

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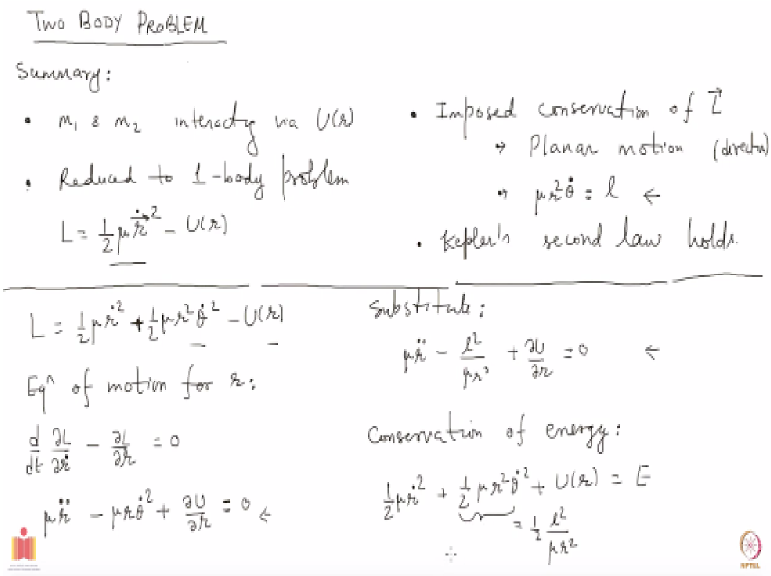

So, let us start today's lecture. Last time we were looking at a two-body problem where we had two particles of mass M1 and M2 which were interacting by a central force which we said is given by a potential energy function U of r. So, m1 and m2 interacting where potential energy U of r and the fact that it depends only on the separation distance, the magnitude of the separation makes it the central force.

And then we saw that we could reduce this two body problem to a one body problem. So, we reduced to one body problem and we wrote down the Lagrangian as L equals half mu where mu is the reduced mass of the system r vector dot square minus U of the potential energy which you had origin in the problem then we imposed conservation of momentum angular momentum.

And by using only the fact that conservation of L implies two things first that the direction of L is going to be fixed it is not going to change and second that the magnitude is not going to change. So, using only the first thing that the direction does not change we could say that the motion happens in a plane the motion will be planer this followed only from the direction then when we imposed or we could utilize the constancy of the magnitude of L in writing down the following that is the angular momentum which will be constant.

And then we could also show that for a central force field the Kepler's second law holds true and then let us proceed from here. I think let me see here it is already we have talked so I can just write down. So, now let us continue with today's next steps. Now because the motion is happening in a plane which I have established based on the conservation of angular momentum I can write down the Lagrangian of the system to be the following.

So, I have L now this half mu r dot square this r is because it is going to be in a plane I can write this as half mu I am writing in now polar coordinates r dot square where r is the radial coordinate and then I should have a term coming from the theta coordinate which is half mu r square theta dot square and then you have the potential energy term. Now, from here we can immediately write down the equations of motion.

So, let us write down equation of motion for the r coordinate so we have to calculate d over dt del L over del r dot minus del L over del r and this will be 0. So, if you calculate these you will find the following. So, you get a del L over del r dot we will give you a mu r dot it gives a factor of 2 which cancels here and when you take a d over dt it gives you mu r double dot then you have del L over del r and both these two terms have r dependence.

So, from here you will get so we have minus which brings a minus here r because its square gives a 2, 2 cancels you have mu r theta dot square then the minus sign will cancel this minus sign and you will get a derivative of U. I can write dU over dr instead of partial derivative as well because U depends only on r so there is no distinction between a partial derivative and a total derivative here.

So, now we can substitute what we had for theta this equation and here and obtain the following. So, we substitute and we get mu r double dot minus l square over mu r cube plus delta U over delta r equal to 0. So, I have removed the theta dot from this equation of motion and now you see this is purely a one-dimensional equation of motion. This is equation of motion of a system with one degree of freedom and that is only r.

The real system is still moving in a plane so it is described by theta, but if you look at the r coordinate it is like a one dimensional system and we have already talked about one dimensional systems and how to solve them in general. So, we can repeat the same steps and if you recall it is easier to start with the first integral which gives you the energy and from there you can make several observations and that is what I am going to do now.

And this exactly what we did when we were talking about one dimensional system. So, let us use conservation of energy and what the conservation of energy says that the kinetic energy plus the potential energy is equal to the total energy and the kinetic energy here is half mu r dot square, but now I want to write it using the radial and polar coordinates and I also want to use this thing this fact so I can write half mu r dot square plus this piece.

See this entire thing is the kinetic energy so that thing and I substitute for theta dot to be this piece. So, I get half mu r dot square and maybe I should write half mu r square theta dot square plus U of r is the total energy and this is constant as I have repeatedly said that this is determined by the initial conditions. Now, this piece is equal to half l square over mu r square this term. So, with that I can write the following. I can write my equation of energy conservation in the following manner.

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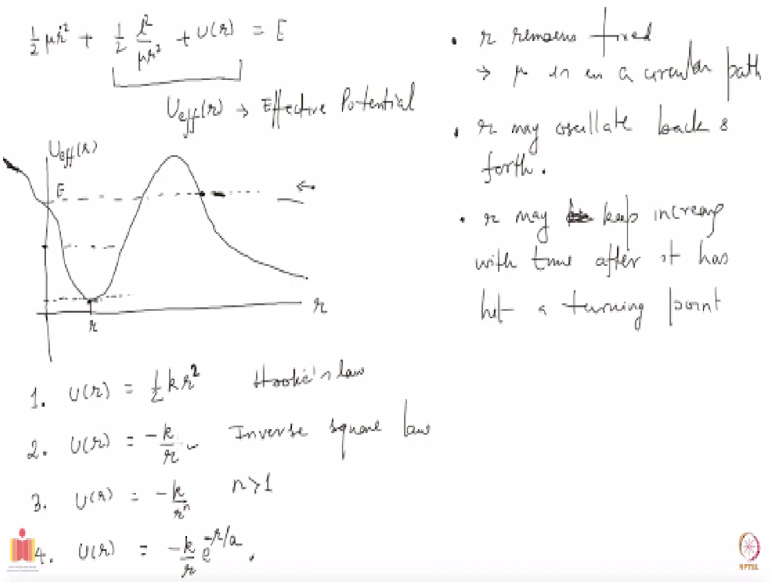

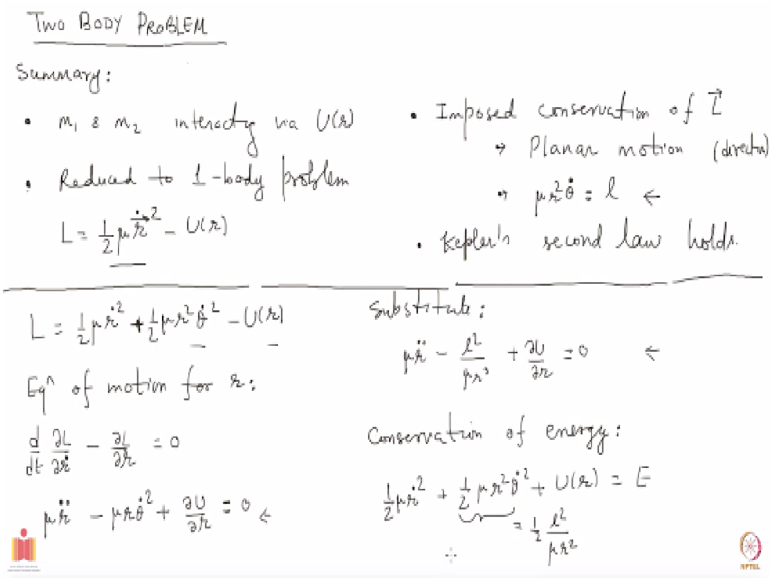

So, I write half mu r dot square plus half l square over mu r square plus U of r is equal to the total energy. This is the centrifugal term here. Now, I will define these two the sum to be U effective I call it the effective potential. So, you understand right why I call U effective because effectively the potential now is the sum of these two and not just U if you look it from the point of view of the coordinate r that is why I give it a name U effective.

And we know what we should do to analyze this problem we remember last time when we were talking about the one dimensional system we plotted U of r now in case of U of r we have U effective of r and here you have r and in general your potential could be anything like this, this is the most general thing that can happen. It could have some minimum somewhere, somewhere a maximum and it could keep decreasing and go to 0 as it goes to infinity. It could keep increasing somewhere in some region.

So, these are the most general features that can be present in the U effective of r and we can now immediately see what are the possibilities that can happen. One is that your system is somewhere here meaning your energy of your system is such that your particle mu is here in one dimensional picture which we have which would mean that the radius or the separation of this particle mu from the origin remains fixed.

So, the particle always stays here which would mean that the particle is going to move in a circular path. See the theta is still going to change with time our theta dot was not 0 let us go back here. So, you had a non zero theta dot so theta is going to change with time so the particle is moving in plane with theta changing, but radius never changes which means it is going to move in a circle so that is one possibility that can happen.

Then what you can have is that you start your particle somewhere here and it comes let us say the particle has energy equal to this value. If your particle has energy this much then if it has started from this direction it is coming it will hit the turning point and return back and it will go to infinity and would never return again okay that is another possibility. If the energy of the system is somewhere here you know what will happen, it will keep oscillating back and forth in here.

And this oscillation is in the r, but while r is shuttling back and forth between in this region the theta is still changing, so your particle is moving in plane with theta changing and the radius is also changing with time this is another possibility that can happen. So, let me write down one possibility is that r remains fixed which would mean that the mu is in a circular orbit is in a circular path or the r may oscillate back and forth.

But this does not mean that particle is moving in a line it just means that in the r direction it is its r values increasing and decreasing with time, but at the same time it is also moving in the theta direction. So, it is not just on oscillation in one dimension that is not the case and third is r may keep increasing with time after it has hit a turning point which was the case here for this one and this is what in general will happen.

Now, in order that we say more we need to know the form of the potential and then we can analyze the motion in more detail. So, we are immediately faced with asking what are the different center of potential that are possible. While there is an infinity of potentials that we can write, but some of them are fairly simple and common. So, you could have a center of potential which looks like half k r square.

Now, this is your Hooke's law which holds true for small displacements in general, but let us say I say that this holds true for all r. Clearly, it is an unphysical situation because as r is increasing the potential will become infinite it will have infinite energy. So, it is not a physical situation, but as far as mathematical function is concerned this is good. So, this is our Hooke's law and this is spherically symmetric potential now.

So, it is a spherically symmetrical oscillator then you can think of inverse square laws so where the forces are decreasing as 1 over r square which means that the potential is decreasing as 1 over r this will correspond to for example gravitational potential or Coulomb potential then you may think that okay instead of having an r here I can think of a higher power of r below and you may ask what happens in those cases.

Another I am just listing down few examples so that we keep in mind that there are several different central of potential that can be imagined. So, you can have a k over r which is same as the inverse square law, but the inverse square law you see this force dies off very slowly. So, it can be felt at very long distances. Let us say for some reason I want to have a force which for small distances is more like inverse square law.

But as the distance increases it dies off very fast. So, you can supply a exponential damping here like this where a will have the dimension of r and you will see that this cuts off the force very fast. So, what you can do is you can compare this force with that force and ask how much smaller this one is if you go to let us say r equal to 10 times a see a in the length scale in this problem.

So, let us say you are 10 times at a distance 10 times a then you compare how much smaller this potential is compared to this. This will give you a quick idea of why we say that this potential cut off very fast as you go away from a at a larger distance. So, these are some of the examples of potential that are possible and you can construct any combination of these and several others you can create maximum and minimum. You can create a central of potential which has maximum and also so there are lots of possibilities. Next now before I start discussing one of these potentials which will be the inverse square law basically.

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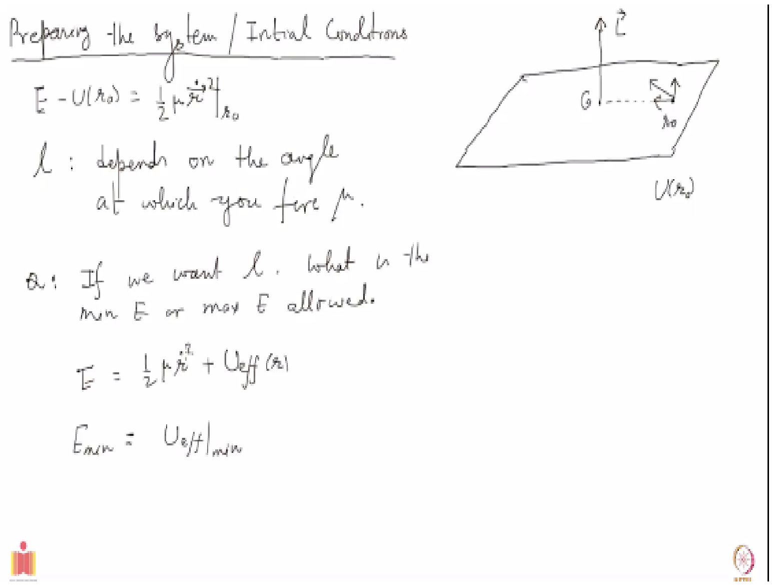

I want to talk a little bit about how we prepare the system I mean if it is in your hands, so or equivalently the initial conditions. So, either you prepare or you are not prepared either way you have some set of initial conditions. So, let us say we are looking at this one body problem to which we have reduced our two-body problem and I want to have my system to have a total energy E some number which I choose.

So, the situation is this we have a plane in which the particle will be moving and clearly I have chosen a plane which means that my angular momentum will be in this direction some L. I have not chosen L yet, but I know the direction I have chosen and let us say this is the origin O where r is equal to 0 and let us say I want to start here some point r let us call it r naught and I want my system to have a total energy E.

Now, if it is here it will have some potential energy U of r0 and now if I want to have this number E as the total energy of the system and I am saying that I am going to start my particle here it already has a potential energy U of r0 and this difference gives me half mu r dot square. So, pay attention I have written r vector not just r dot and this at r0. So, which means that I know or I can calculate from here with what velocity I should fire the particle at r0.

So, if I calculate this find the number the velocity from this and I should fire my particle with the calculated amount of velocity. If I do so then my system is going to have the total energy E which I wanted, but I have freedom in choosing the direction in which I fire the particle so

I could choose to fire it towards the origin. If I do so the energy is still E, but the angular momentum will be 0 or am I choose to fire its by giving it a small angle.

Or am I choose to fire perpendicular to this radial direction and in that case I will have the maximum angular momentum that is possible with that velocity. So, depending in which direction you are going to fire you will get whatever value of angular momentum consistent with that. So, in that manner you can prepare your system to have whatever value E you want and whatever angular momentum you want.

And anyway by choosing this plane in which you are firing you have already decided on the direction of the angular momentum. So, given E several possibilities are there for the magnitude of l depending on at what angle you are firing here, that is good. Now, let me just write it depends on angle at which you fire the particle mu. (())(26:07) Now, you may ask if I choose some value of the angular momentum that I want for my system.

So, I wanted to have some angular momentum L what is the minimum energy or maximum energy that is allowed for the system. So, we ask if we want angular momentum to have value l what are the minimum, what is the minimum energy or maximum energy allowed. So, E could be anything and the upper limit is not bounded because even if you have very large energy you may fire it very close I mean if you can fire it almost in the direction of origin.

If you do so then you can make the angular momentum small okay that way so you can offset the largeness of the energy to make small l, but the lower limit is of course it is there is a lower limit on the energy and that is what we want to see how. So, the lower limit is this you have E the total energy E is equal to half mu r dot square not the vector plus U effective. You remember U effective has two pieces.

One is the potential energy of this interaction plus the centrifugal term coming from the motion in the theta direction. Let me just show you once more here. This is the U effective and also note that the this piece which is coming here is positive it is always positive, but U of r this could be negative also because there may be attraction. So, there is a l dependence in here in U effective and let us say you have chosen some l.

We want to know what is the minimum total energy that is allowed for the system to have and clearly the minimum energy this can have is the minimum of this provided I can put the r dot to be 0. So, the E minimum you get the E minimum to be equal to U effective minimum.

So, if your particle comes to not rest because rest would mean that particle is not moving, but r dot equal to 0 does not necessarily mean that the particle is coming to rest.

It just means that the particle has come to a turning point. Let me emphasize this so you see here if the particle is let us say coming from this direction and going when it reaches here the r dot will be 0 because it has to turn now and it does not mean that the particle has come to really at rest because in the plane it will still have some theta dot non zero so these two are different things.

So, anyhow the minimum energy that the particle can acquire is equal to the minimum value of U effective. It cannot go lower than that because going lower than that would mean that this piece has to become negative which is not possible because it is a square of real number which will be positive. So, that is the constraint we have. Now, with this understanding let us look at Kepler problem now which is basically taking a inverse square law force which is applicable for gravitational force for example and that is what we will study now.

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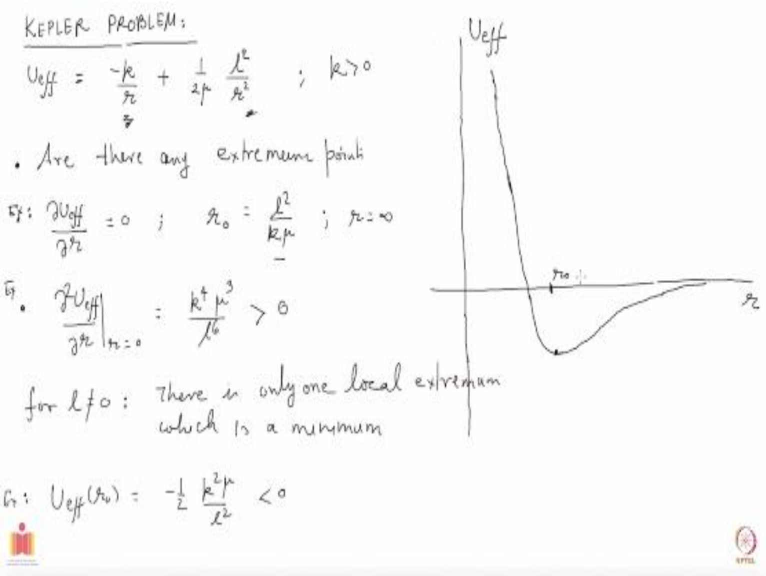

So, let us write down U effective, U effective would be U of r so U of r is now minus k over r then you have the other term coming from centrifugal term 2 mu l square over r square. I would encourage you to think why you have such a term I mean of course mathematically it is all good that it is there, but you should develop a feeling for this term that yes indeed you really see that it should be there.

For example you can imagine sitting on the one way to think will be to sit on the radial vector let us say you are the observer which is sitting on the radial vector which connects to the point mu the particle mu and you have been told that this particle mu is getting attracted towards the center by this thing, but as the particle is moving around you will not see that this is really attracted by this force.

You may see it having additional force and that will be due to the fact that this thing is really moving in a circle sorry moving in a plane. So, just imagine that this thing was moving in a circle and you are sitting on that radial vector then what you will see is that even though you have been told that there is a force which is attracting it that particle does not move anywhere it stays exactly at the fixed distance away from you.

So, for you nothing is happening it is just staying there and the reason is that actually it is moving, but it is the motion is in the theta direction. So, from your point of view you will think that there is another force which is balancing that attraction towards the center and there is another fictitious force and which is this one. So, anyhow that is our effective potential and to say anything about the motion we should draw a diagram like this for the Kepler problem.

So, the first thing you should ask is are there any maxima or minima so what is the structure of U effective. So, first thing is let us find are there any extremum points and if they are there weather there is a minimum or maximum or if there are many maxima or many minima or both kinds are present that is what you should ask. So, you can check immediately the following that if you take the first derivative and equate to 0 you get a non-zero.

I mean you do get a solution and this happens at r naught equals l square over k mu. So, there is a potential candidate for an extremum please check this then also let us calculate the second derivative to know the nature of it. So, if you calculate this quantity at r equal to r naught you will find that this turns out to be I hope it is correct k to the 4 mu cube over l 6 yeah it is correct I think l to the 6 which is greater than 0.

Your k is greater than 0 I should have written specified here k is greater than 0 anyway mu is greater than 0 l is l square so it is positive you have l cube so it will be a positive. So, this is a positive quantity which means that at r naught you have a minimum. The other solution which you will find here corresponds to r equal to infinity and of course at r equal to infinity you do not have any extremum it is not a local extremum it is just falling off to 0.

So, you have one and only one extremum and that is a minimum in this case. So, for l equal to 0 there is see in this derivation I have choose taken l to be written on 0. So, for l not equal to 0 there is only one local extremum which is a minimum. Now, you know that a point r naught which is this you will have a minimum then you know that your U effective goes to 0 asymptotically as r goes to infinity.

Then we also know what do we know yes that as r goes towards 0 your potential will blow up and it will blow up in the positive direction because this will blow faster than this one. So, for very small r really towards 0 you can forget this one and this one will be the one which will control the behavior of U effective. So, this is the kind of graph you expect for U effective for the Kepler problem.

Before that I should say something more I should also give you an exercise trivial that find out what is the value of U effective at r equal to r naught where the minimum is located. So, U effective at r0 is equal to minus half k square mu over l square which is negative. So, now I know quite a lot so let us say this is r0 let us say this is the value of U effective at this point which is negative. So, that is why I have chosen this point to be here.

And then it goes asymptotically to 0 because at r goes to infinity this goes to 0 so I should have something like this the slope should be 0 here. It should go to 0 not looking very nice I will do it again I hope it looks better and then it blows up. So, your y axis is U effective this is r and that is the effective potential that u have. Now, our task is to analyze the motion of the problem, motion of this particle mu in this potential that is what we will take up next.