Introduction to Classical Mechanics
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Lecture 30
Two Body Problem, Kepler’s Second Law

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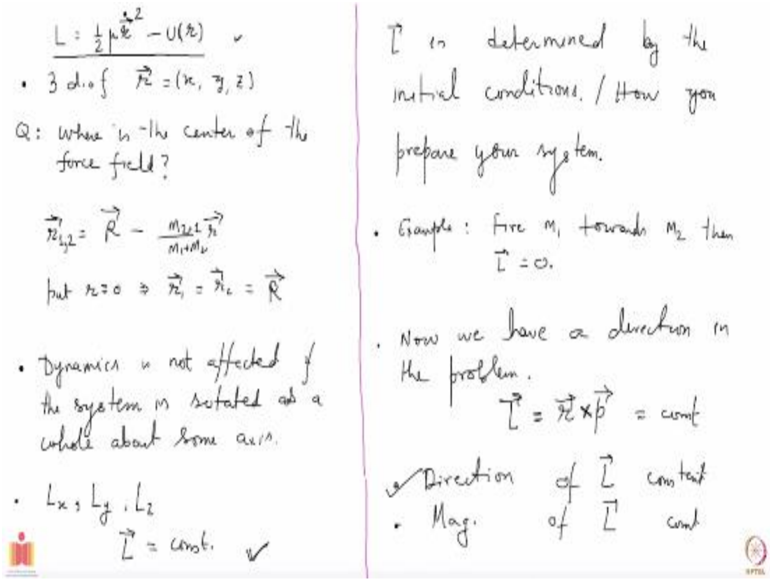

So, we have reduced our two body problem to a one body problem of mass mu a particle of mass mu that is moving around in a center of potential given by U of r and we have not yet specified what U is other than that this is a central force field. Now, this looks like a problem of 3 degrees of freedom. So, we have gone from 6 degrees of freedom to 3 degrees of freedom and because your r is now x, y, z coordinates x, y and z.

And you know what r was that is the separation between the two particles with mass m1 and m2 and you may ask where is r equal to 0, where is the origin of your potential field, where is the center of the force field that is easy to find out where it is. Let us recall what were the expressions which were relating r1 and r2 to capital R and small r. If you recall we had r1 to be this was the center of mass radius vector for the center of mass.

And then you had this piece and for r2 you had the corresponding thing. Here it was instead of 2 you had 1 here otherwise it was the same thing. So, clearly if you put small r to be 0 r1 and r2 both become capital R. so that is the location of center of mass. So, your r equal to 0 corresponds to your both the particles being situated at the center of mass. The place, the center of mass is where the r equal to 0 so that is fine.

Now, let us go back to our discussion of like Lagrange which you have written down. I was saying that let us ask whether these are the best coordinates to use the x, y, z and clearly, they are not because your system has a spherical symmetry so you would like to use more suited coordinates here. So, first point to note before we can make some progresses that because we have a central field meaning the only radial separation methods.

Maybe I should make this point that often I would be switching back and forth between the interpretation of the two body system as one body system which is given here as the Lagrange right now or sometimes I will be making statements which will pertain to picture where really those two bodies are present, so I hope it will not be a issue in understanding. So, because our potential energy is dependent only on the coordinates small r only the magnitude of it.

It does not matter at what Azimuthal or polar angle that point mu is located meaning if I take my system and rotate it by some amount about some axis and that will appear as a cyclic coordinate you know that is a symmetry of the system and because you know there are three independent directions let us say three mutually perpendicular directions about which you can do independent rotations.

You will have correspondingly three conserved quantities, three conserved angular momentum which are Lx, Ly and Lz this is what we discussed very much in the beginning of this course. So, let me write it down here the dynamics is not affected at all if the system as a whole is rotated about some axis. Now, as I said because you have three directions mutually perpendicular directions you can take.

This will imply that there are three angular momentum which are conserved which are Lx, Ly and Lz or equivalently I can put it as a vector equation and say that L equal to constant. This will be a constant of motion so whatever your system does L will remain unchanged. So, this is what the symmetry implies, but of course this will not tell whatever information is given to you cannot tell what the value of this vector L would be.

So, L cannot be determined from whatever we have talked till now because that is going to be determined by the initial conditions or by how you prepare your system. So, L is determined by the initial conditions. I hope this is clear which is same as saying how you prepare your system. I will give you a simple example to understand this. So, you have your so let us look at your two-body problem so let us think of both the particles M1 and M2 are there in front of you.

And let us say particle number M1 is located at some place and you take particle number 2 which interacts with particle number 1 by a central force meaning the forces dependent only on the separation and you take particle number 2 and fire it directly towards particle number 1. Now, if you do so the angular momentum would be 0 this (())(08:33) should be immediately realized just by from the definition of angular momentum.

That the angular momentum about the center of mass or about r equal to 0 this will be 0 of this system this particle we will be following or going towards directly the particle number 1. So, I have prepared my system such that the angular momentum is 0. Now, I could fire particle number 2 at some angle rather than making it directly go towards M1 I can fire it at some angle and give it some velocity that way I can change the direction. And the magnitude of the angular momentum which the system is going to have.

So, your L is controlled by the initial conditions. So, let me just write down here what I said example if you fire M1 towards M2 then the vector L would be 0 that is fine let us see what else I want to say. Now, let us go back to our Lagrangian here and try to see what good coordinate I can use.

Now, the moment I say there is an angular momentum of the system now I have a direction in the problem. The direction of L is now we have direction in the problem. So, till now in the discussion I was always saying the potential is spherically symmetric and of course clearly there are no directions, but now we realize that there is a direction in the problem and that is given by the angular momentum.

And let us see how I can utilize this two make further remarks about the statement. So, I found a direction to the problem and this is clearly your L is r cross p. This r is measured from the origin and you know where the origin is so angular momentum is measured with respect to that point. So, this equation says that because this is constant for my system this I have already established here based on the symmetry of the problem.

This says something nice. First of all you observe that because L is not going to change with time its direction is not going to change with time neither its magnitude both the things will be constant. So, let me write down direction of L constant and magnitude of L is also constant. So, let us see what we can see from the fact that the direction is constant. So, because this is a cross product.

You know that L is perpendicular to r or r is perpendicular to L either way which means the particle of mass mu is moving in a plane that is perpendicular to L. See the r has to be always perpendicular to L which means the r is confined to a plane perpendicular to L and of course it has to have also momentum within that plane. So, what I have been able to show is that because the direction is constant of L my symmetry dictates that the motion will happen in a plane. So, these two particles will always remain in a plane so that is what our conclusion is.

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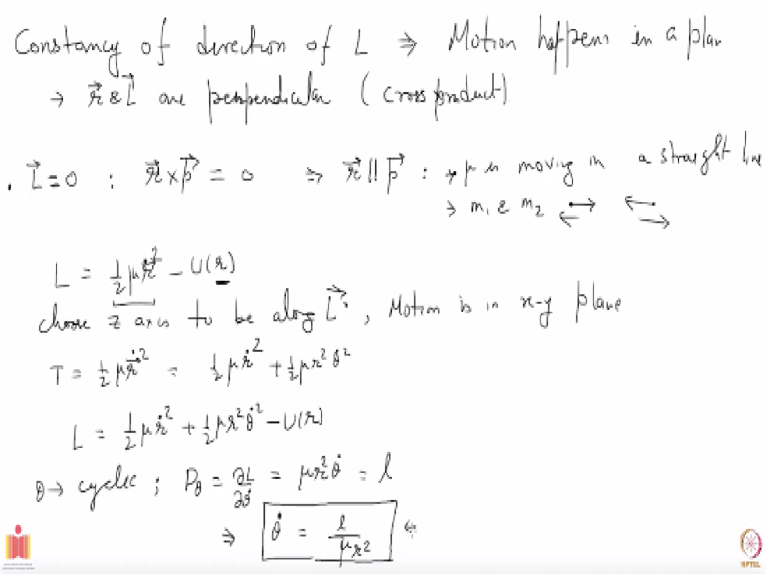

Constancy of directions of L implies motion happens in a plane and the reason as I said because r and L have to be perpendicular and that is because of the cross product definition. Okay that is nice, what if L is equal to 0 then I do not have a direction because L is 0. In this case, my r cross p is 0 and that will happen to be 0 only if r and p are in the same direction if that is the case then this cross product will vanish.

And your particle is this is moving in a straight line or equivalently the two masses are directly falling into each other or just getting separated from each other along the straight line that is what I said a few moments ago. Let me write it down so r is parallel to p so your mu is moving in straight line or for our I mean the picture of both the bodies present for that it will mean that m1 and m2 are going towards each other this way or that way along the straight line, so this is good.

So, now that I have established that the motion is happening in a plane I would like to write down my Lagrangian which was here using this fact. So, let me write down L is half mu r dot square minus mu or r. Now, I know the particle is in the plane you can say that I will align my z axis with the direction of L and the plane in which the motion is going to happen would be x, y plane that is one choice with you that is the choice you can make without any loss of generality I can just choose z direction to be the direction of L.

So, I will say that so choose z axis to be along L and the motion happens then x, y plane. Now, I can write down this kinetic term half mu r dot square in the x, y plane using polar coordinate because that will be useful you have radial symmetry here. So, instead of using Cartesian coordinates x and y it make sense to use the polar coordinates. So, this becomes half mu r dot square let me put a vector here r dot square plus half mu r square theta dot square.

And then you have the potential term here. So, my Lagrangian now is I should have anyway let me write it down again half mu r dot square plus 1/2 mu r square theta dot square minus U of r. Okay this r is just the radial distance it is no more a vector because we also have theta now and this is the Lagrangian and we should always look whether something is cyclic here and indeed it is.

You see that theta does not appear anywhere the r does appear of course everywhere, but theta does not which means theta is cyclic and which further means that the momentum conjugate to theta would be conserved and you know theta is the polar angle so there is nothing polar it is all in the plane. So, it is the angle so it means that the corresponding angular momentum will be conserved.

Which is you can find P theta just del L over del theta dot which you already know that it is angular momentum you get taking the derivative mu r square theta dot and this is a constant and that constant is of course going to be determined by your initial conditions and let us call that L and you can clearly see that this is just the angular momentum of your system or magnitude of capital L magnitude of this L here.

So, you have got a first integral here which involves only one derivative and this is what you expected you would have got anyway the conserve quantity here and this small l is determined by the initial condition as we discussed sometime back. Now, let us make some before I do that let me just write here this implies that theta dot is equal to l over mu r square. So, I have solved not really because it is still r is still present there. But anyway I have such a relation now. Now, I want to make a quick interpretation of what I have got, what this condition of conservation of angular momentum here in terms of this is specially implies.

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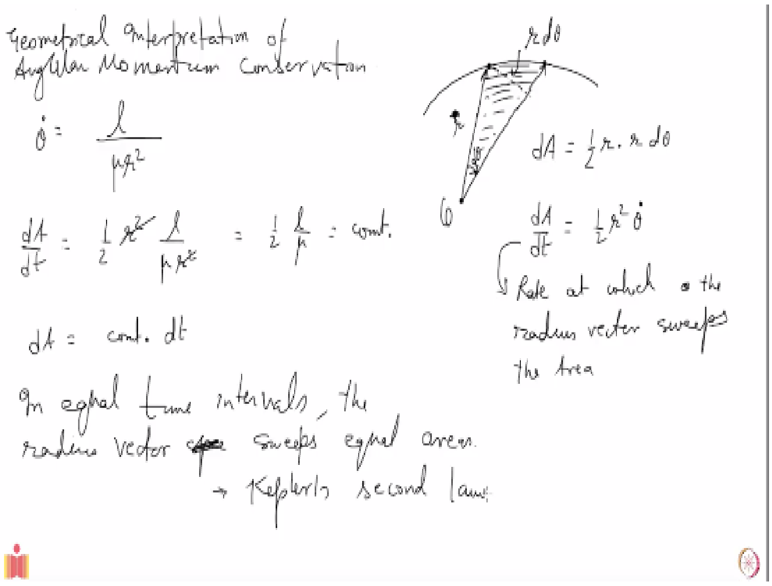

So, I am looking for a geometrical interpretation angular momentum conservation. So, I want to see what it would geometrically mean if someone tells me that you have a central force problem and your angular momentum is conserved the motion is happening in one plane. All this would amount to what geometrical feature of the problem. So, let us say this is the origin of my radial coordinate.

This small r and this plane of the screen is the plane in which the motion is happening and your particle mu is moving in some way. At some point it is here and this is the radial coordinates so this is the radius vector r after some small infinitesimal time t sorry dt what happens it moves here and it covers an angle d theta. Let me draw a perpendicular here and this arc length is this is r it is r the radius vector.

This is r d theta and if you find out the area of this infinitesimal sector. You see if this is infinitesimal, so this area which you are seeing here is this half r into rd theta that is the area of the triangle or the sector which coincide when your angle is infinitesimal time and if I calculate dA over dt I all I have to do is divide both sides with dt I get half r square theta dot. So, this is the area the rate at which so this quantity is the rate at which the radius vector sweeps the area.

So, when this vector is moving okay this particle is moving the radius vector is moving it is sweeping some area and this gives the rate at which that area is swept. Let us see what our angular momentum conservation implies about this rate of this dA over dt. So, here I had found theta dot to be l over mu r square so theta dot was l which is a constant over mu r square.

If I substitute in dA over dt I get dA over dt to be half r square and theta dot is l over mu r square and the r dependence goes away it cancels and you have half l over mu which is a constant. So, what it says is that the amount of area that the radius vector is going to sweep in some time interval is same as it would do at any later time if you take again equal time interval.

So, if you take I mean I can write in this way dA is this constant which you have found half l over mu into dt. So, area in equal times intervals the radius vector sweeps equal areas. Okay this is nice we have determined this which is the famous Kepler's Second Law. This Kepler Second Law we have determined based only on the fact that angular momentum is conserved and the forces are central in this case in this discussion beyond that I have not used anything.

So, just to summarize before we move next, we have found two things. Motion happens in a plane and we have found Kepler's Second Law holds true these two things and I have assume nothing more than the fact that I am using central forces. Now, let us look at the equations of motion.