Introduction to Classical Mechanics
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Lecture 29
Two-Body Problem

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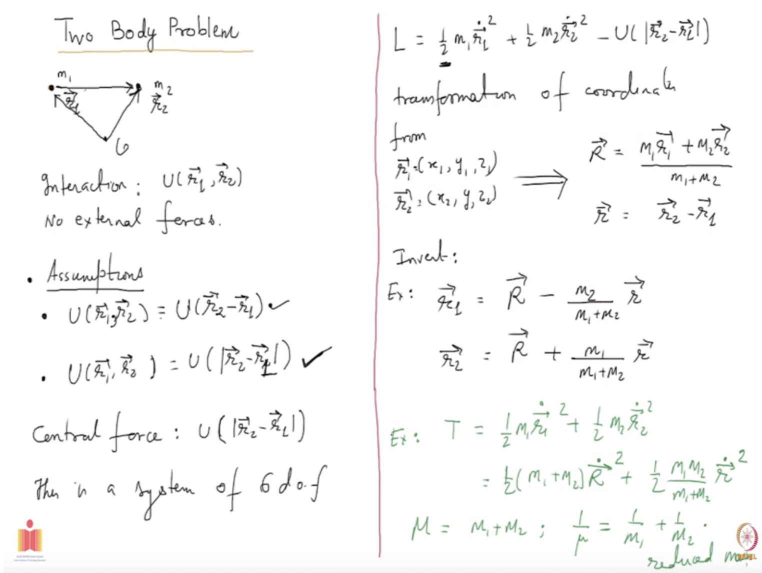

Today we will start looking at, what is called Two Body Problem. Here you have two particles of masses m1 and m2, which are interacting with each other and there are no external forces present, so that is the situation. So, you have a particle of mass m1, another particle of mass m2 and let us say this guy is located at r1, and this particle is located at r2, with respect to some origin.

Let us say it is here, which means this is r1, and this is r2. And these two particles are interacting with each other and let us say the potential energy due to that interaction is U of r1 r2. So, the interaction is given by the potential energy r1 r2, it is a function of r1 and r2 and there are no external forces on this system.

So, this is what is a two-body problem, so you have two particles, two bodies and they are interacting in a certain way. So here, I have made an assumption that the potential energy only depends on where the particles are, that is the assumption I have made. But now I want to further assume, which is very reasonable, further assumptions.

So, I assume that the force is such that the potential energy does not change if you take the entire system and move it to somewhere else. Meaning, it cares only about what is the difference between r2 and r1. So, if I translate this entire thing by some amount it, the potential energy does not change. So, I am saying that U of r1 r2 is same as U of r, let us, let me put the first one here in this, does not matter. So, here what I am saying is, it cares only about the vector which is separating them, the, the difference vector r2 minus r1.

That is nice and that is of course is not unreasonable, this is a very reasonable assumption because that is how most of the things will behave. If for example, you think of two particles which are interacting gravitationally. If you take the entire system and shift it to another place, millions of kilometre away or billions of kilometre away the potential energy between those two will not change. Now not only that, we want to further assume that U of r1 r2 is only dependent on the distance between these two particles, so this is clearly different from this.

Here, it is still depended on the direction in which r2 minus r1 will point. So here, let us say, this is r2, this is r1, so this vector. This is r2 minus r1, so when I write here this way, it means that the potential energy cares about in which direction this vector is pointing. So, if you rotate this, for example this entire set up then the r2 minus r1 will be pointing in some other direction and it, it says that okay it may depend on it. And here I am going to be more specific in saying that, even if you rotate this so that the direction of r2 minus r1 changes, it still does not change if the separation distance is fixed.

That is what is meant by this expression, which is also not unreasonable. This is a very reasonable choice because you know, that space is homogeneous and isotropic. So, if I, if my system is not being affected by any other external agents, then space will be homogeneous and isotropic and translating as a whole the entire thing or even rotating the entire setup should not change how the system is going to behave which means that the potential energies should be given by such restricted form.

Let us see, now if this is the case then we say that the force is central. Central force, which means just that your potential energy depends only on the separation between the two particles of your system, that is a central force field. Now let us ask, and we would like to solve this system and this system is solvable, we will see that we will be able to find the full solution to this. But at the moment the goal is to figure out as much we can about the system without specifying anything about the potential, other than what we have already said.

So, I do not want to assume any special kind of, special form for the potential that it falls off this way or that way and I want to see how much I can say about a such a system just based on the fact that this is a centre force system. So, one thing is clear that this is a 6-dimensional system, this has a 6 degrees of freedom, which is obvious each particle has 3 degrees of freedom. So, the system is, is a system of 6 degrees of freedom and let me try to create a separation. So, what is the lagrangian of this system, it is easy to write, it is just the kinetic energy minus the potential energy and we have already written on the form of potential energy.

So, your L the lagrangian is half m1 r1 dot square plus half m2 r2 dot square minus the potential energy, which is r2 minus r1, the modulus of it. It cares only about the length. Now, we ask whether these are the best coordinates to describe the system. Now, our experience with several different systems before may immediately tell us that they, these are not the best coordinates.

It would be wise to use r2 minus r1 the modulus of it as one of the coordinates, and the other coordinate which will be good to use will be the centre of mass coordinate. We have come across this thing several times, so I will not go further into this and just write down the transformation. So, what we want to do is, we want to do a transformation of coordinates, of coordinates from r1 which is, I will call x1, y1, and z1 and r2, which is x2, y2, and z2.

And from this I want to go to R, Capital R by which I denote the centre of mass m1 r1 plus m2 r2 over the total mass of this system, which is m1 plus m2 and small r, which is r2 minus r1. So, I go from this 6, 1, 2, 3, 4, 5, 6 to these 6 because r is of 3 vectors, it has 3 components and small r is also 3 vectors, it has 3 components, so I have gone from 6 to 6 and let us see why this will be very useful.

But before do we do that, let us invert this and write down r1 and r2 in terms of capital R and small r. So, please do this exercise, so please check that what I am writing is correct. r1 you can write as the centre of mass, location of the centre of mass minus m2 over m1 plus m2 and this small vector r which is the separation vector, and then your r2 is this plus m1 over m1 plus m2 times r.

That is nice, and this will be very useful because when I try to find out what the kinetic energy is, which is here, this place. And when I substitute these r1 and r2 in there, I find a very nice result, which is kind of expected. So, please do the following exercise and check. I had a plan of using green when I am giving exercises.

So, please check that, the kinetic energy T becomes half m, kinetic energy which is half m1 r1 dot square plus half m2 r2 dot square becomes half m1 plus m2 r dot square, which is the coordinate of the centre of mass so the velocity of centre of mass square plus half m1 m2 over m1 plus m2 r dot square.

Let me check the dimensions, so I have 2 powers of mass here, 1 power of mass below so it is, it makes 1 power which is consistent with this one so looks fine. Now, let me define, by capital M M1 plus M2 the total mass and I also introduced mu to be the following, so that is the definition of mu. And you can see that this piece here, M1 M2 over M1 plus M2 is just mu, and mu is called the reduced mass.

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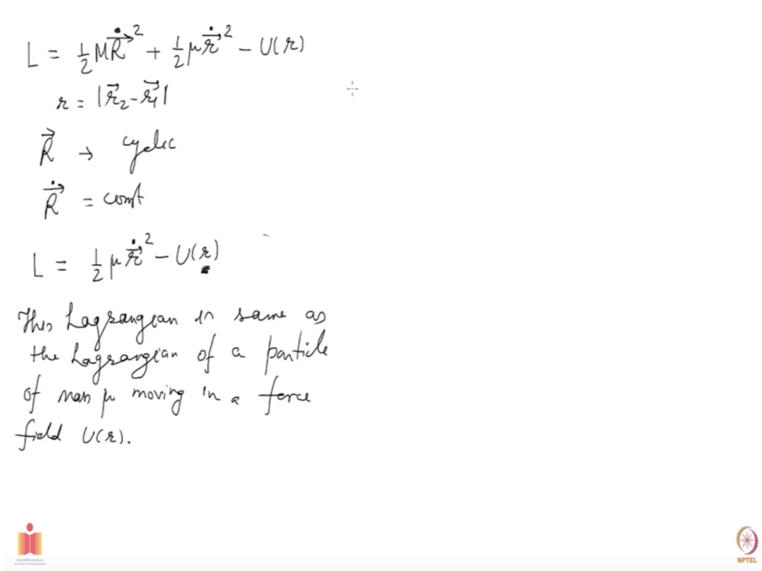

So, with this what is our lagrangian now, our lagrangian looks like the following. So, my lagrangian is now, half M R dot square, then you have the reduced mass term which is half mu r dot square and then we had the potential term which is U of R, where I have introduced new notation which is r2 minus r1. So, this vector, this r without any vector symbol is this quantity.

That is good, and it is nice because you see the coordinate capital R, the centre of mass coordinate that is a cyclic coordinate which is what you expected.

This, this is nice because now I can immediately solve the equation of motion for capital R and I know what I will get, I mean you can do it but what, that is what you are going to get the equation of motion for R is R dot is equal to constant, which just says that the centre of mass moves with a constant velocity that is what is expected.

And then we can drop it from the lagrangian because the equations of motion of other coordinate R is not going to be affected by what is happening to capital R. So, I can take my lagrangian to be half mu r dot square minus U of r. Now, you see this is very nice because this looks like, I mean not like it is exactly of the same form as of a particle of mass mu which is moving in a potential field, which is given by U of r, central potential field. So, this lagrangian is same as the lagrangian of as the lagrangian of a particle of mass mu, mass mu moving in a force field U of r.

So, you understand what is happening, you started with a system which had two particles but with a good choice of coordinates the system appears to be equivalent to another system which is a very simple system where you have a particle of mass mu which is moving in a central force field, which is given by U of r, that is what it looks like.

So, if you can solve this system, the this new one to which we have reduced the our original system if we can solve it, I can invert back. So, let us say I solve for small r, and I have capital R, from that I can construct my r1 and r2 and I will know about my original problem. So, that is what will provide us the full solution to this problem.

So, we will now proceed from here and ask, whether I can further make a better choice of coordinates. I went from r1 and r2 to capital R and small r, which was centre of mass and the separation between the two particles, which immediately gave me a cyclic coordinate and I could get rid of that coordinate from my, from my lagrangian.

I would like to ask, whether there is still something I can do about the, the coordinates, can I choose a nice coordinate and, and you can see why I might be thinking about this because you have here only the, the magnitude of that vector r, so I can, I can do more so I will, this is what we will look at next.