Introduction to Classical Mechanics
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Lecture 28
One Dimensional Systems

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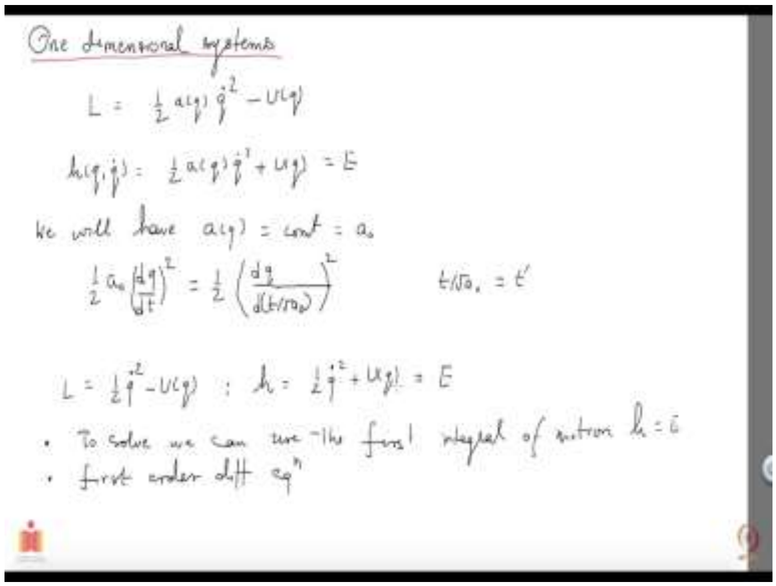

We have studied a particular one dimensional system that is the Harmonic Oscillator in some detail in the last several videos. But here we want to talk about One Dimensional Systems in general. So I am not going to assume anything specific to given one dimensional systems. I am going to treat it to be a fairly general and we will try to see what all we can say in general for one dimensional systems.

So, that is the task we have in front of us. So, if I want to write down the Lagrangian of a one dimensional system, which let us say is which I am going to label by a coordinate q, the generalized coordinate q, then the Lagrangian would be as you are already familiar would be the following.

So, L will be the Lagrangians half a of q, q dot square minus the potential energy, which is a function of q. Remember that there will be a function here which depends on q, this is because you have gone from your Cartesian coordinates to generalize coordinates q. Now, here you see the time does not appear in the Lagrangian though we have written here which means that this will have integral of motion or what we call Jacobi's integral which will be a conserved quantity.

So, that energy functional or Hamilton function that will be a conserved quantity and you can immediately obtain h, I think this is what we call earlier h q q dot and this will turn out to be a of q, q dot square plus U of q and for the system this will be constant throughout the motion and it will be some number E, which we call energy.

Now, frequently we are going to encounter a situation where the a q is a constant which you have seen earlier in our several discussions during studying harmonic oscillators. So, frequently, we were having a of q to be one or some constant. So let me write this frequently often we will encounter a of q to be constant.

Let me call it a naught and what I can do is I can so here I can I will have a naught I can do scaling of time and I can take, here let me see. So, let us do it here. So I have half a0 dq over dt square and if I define I mean I can just write it like this half dq over dt over square root of a naught square that is a same thing as above, perfect.

So, if I define my new time to be like this, then you can see that I will get rid of the a naught. So, under a time scaling I will be able to write the Lagrangian as half q dot square minus U of q, that is good and now your Hamiltonian would be or the energy function would be h will be half q dot square plus U of q perfect.

Now, we want to solves this system in general, but I do not need to set up the equations of motion using Euler Lagrangian equations because I can use this integral of motion and set up a first order differential equation. So let me write down this thing. This is I will to solve this problem we can use first integral of motion and remember this is a first order differential equation. So, let us do that.

So, here I have half q dot square plus you equals E, E is a constant. All I have to do is take U to the other side and take a square root and then I get a expression for q dot which I can integrate, that is what I am going to do now. So as I said, I can write the previous equation, which I had like the following.

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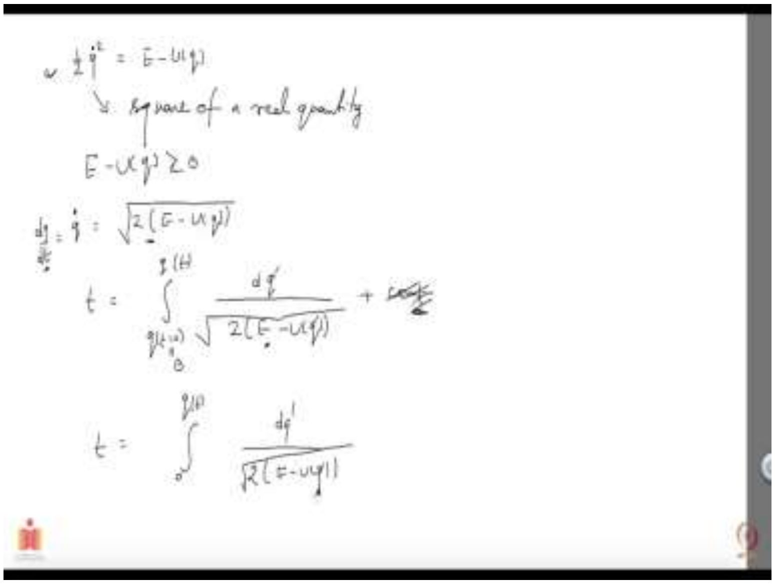

So I can write half q dot square equal to E minus U of q. Note that even before I start look integrating things. Look at the left hand side left hand side is a square of real quantity because q is real. So q dot has to be real which means that this can never be negative.

Which says that the total energy of the system should always be greater than the potential energy that the system has at any q value. I mean if a system is at some q then the system should have a total energy which is more than the potential energy that the system has at that point. So, E minus U of q should always be greater than equal to 0.

So, that is the constraint we have and as you may be aware that this is not necessarily true in quantum mechanics. But classical you see that this has to be obeyed. Now, let us start looking at how we will get a solution. Let me integrate this equation is 1. So I get q dot 2 E minus U of q and that entire thing will be in a square root and I can integrate this so you write this as dq over dt.

So I take dt, when I can jump one step there is no need to write again. So, basically I am put taking dt there and I will bring this expression below dq and I will do integral which will give me t to be dq over 2 E minus U of q square root and there will be a constant of integration that is what you get and let me just put the integration variable to be q prime.

Now, this is a I mean if this this this provides a solution so I can know where the particle is at I mean if I before I say that let us look at the following. You have you expect two constants of integration because your system is really described by Euler Lagrange equation which are second order differential equations. So they should be two constants of integration.

But now we have not really solved those but does not matter whichever we I solve if my question is going to be described by a solution which will have two constants of integration. It should have here also. So, as you may see you already have those two because one is a constant which you have got by doing the integral and another one is E and both of these you are going to control by your initial conditions meaning these are determined by your initial conditions.

So, let me do a little bit more I can write this as so let us say I say that when q is equal to so I do away with this constant. I will do a definite integral now. So I will put q from t equal to 0 to q of t. So this is a definite integral now. And this I want to choose to be 0. Let us call this equal to 0. So what you have basically is the following 0 to q of t dq prime over 2 E minus U of q prime correct.

Now, so if I can calculate this integral if you are given a U of q and you see it is only the potential energy that describes the system. So if you are given a U of q you can do this integral and get t as a function of q. And in principle, you can invert it, it may not be possible always but you can invert it let us say and then you have the trajectory.

But in any case we call that the problem has been solved because we have t as a function of q in principle that is nice but what we will do now is try to say things in general by just looking at the potential energy function. This U of q and as you will see a lot can be said for one dimensional systems without doing any integrals without doing anything, that is my next task.

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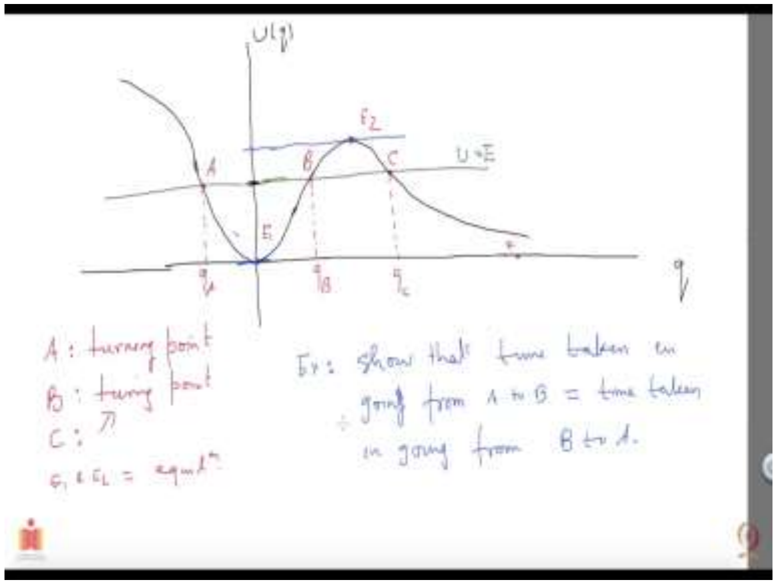

So, let me draw a potential energy graph, what happened? So, along the horizontal axis we have q here on the vertical I have U of q that potential energy and in general what you can have here is that the potential may have some places it becomes it takes a local minimum at some places you can have maximum the function could be going to 0 at the infinity.

These kind of things can happen and there is some most general feature you can put in your potential energy function. So, I am going to draw a graph here which will be depicting a general situation. So, what I do here is I just choose the place where I have a minimum to be at the original. The one of the original one of the minimum of this potential I choose to be at the origin which you can do you can always do so and let us say the potential is not looking very nice.

It is doing something it is goes let us say I do not want to it to be more parabolic. Something of that sort. So, this is one of I mean, this is the most general situation. It has a maximum minimum. You can have maxima here also this you can put several maximum here, but the general feature is this. So, let us say my system is described by such a potential energy and I choose initial conditions such that the total energy of the system E lies somewhere here.

Let us say lies somewhere here. So let me draw a horizontal line here. You will understand why I am doing so, maybe I can do choose some colour. So, this is E very nice and let me also indicate these points. Let me call it qA, qB, qC and I want to label these points as A, B and C. And this point as E1, this point is E2 perfect.

Now, you see let us say you start your system with this total energy E which I specified now somewhere in this region. If you do so, then because E1 is a I mean nothing to do with E1, but let us say you start it here or here does not matter and it is moving let us say the system is moving in this direction.

So, it is q is going towards qA. Now it will have a kinetic energy here because of which it will be able to climb up to the point A and at this place it will run out of its kinetic energy because your potential energy will be equal to the total energy. So your t has to be 0 but there is a force which is still acting on in this direction.

So, the system starts moving backwards. So, this point is called a turning point. So, the point is A is a turning point. So, this system starts rolling backwards goes further here shoots up to B and it again runs out of kinetic energy and it has to turn backwards and so B is another turning point.

So, as you can see that if you have started somewhere between A and B. Then the system is going to oscillate from A to B. It will keep oscillating back and forth between A and B and it will have some time period t which we can determine binding those integrals, but that is what is going to happen. So that is one situation which can happen in a general potential. But let us say you do not start your system here. You start your system to the right of C.

Let us say somewhere here or here and at some such values and let us say your system is moving in this direction. Then what will happen is it will keep climbing up by climbing I am using the analogy with a hill or something, but it keeps climbing up the curve and it when it reaches point C, then it runs out of its kinetic energy because your total energy is going to be U and but still there is a force in this direction.

So it goes there and sees again another turning point and it rolls back towards positive q and it never returns back it just keep going on if nothing is if there is no other maximum there or if there is no maximum which is as big as this line E. So, it will keep rolling down and it will be gone to infinity that is what will happen.

So, your point C is also a turning point, the point E1 and E2 as you already know. These are points of equilibrium one is stable in this case another was unstable but this is all that can happen in this system. So, one it can oscillate back and forth in such a place or it can go here and get lost to infinity that is all.

There are other possibilities which can occur if you choose a different kinetic energy or different total energy for the system. So, let me try to get that one, let us see. Now, suppose you start with the total energy, which is equal to E2. So your total energy is E2 but you have started here. So, you have put the system here where the potential energy itself equal to E2. So, it stays there nothing happens.

If it is not disturbed it stays there forever. So, that is what is going to happen. Similarly if you put the system here with energy equal to in this case 0 because you have chosen the origin to be here. Then it stays there forever. So, these are all possible motions that can occur in a one-dimensional system.

Let us see what more I want to say. I will give you an exercise here. So, what you have to show is the amount of time the system takes in going from A to B will be equal to the time it will take from going from B to A. So, show that time taken in going from A be B is equal to time taken in going from B to A, okay that is what you should show and all you have to do is use the equations of motion you do not have to solve anything and think why this should happen.

From your experience you already know that this will happen, that I mean this should happen but what you have to do is show explicitly from question that indeed this will happen. So, let us proceed further let us ask what will be the time period of oscillation if our system is oscillating between A and B and that is quite simple to say not happened something not very nice. Let us add one more sheet.

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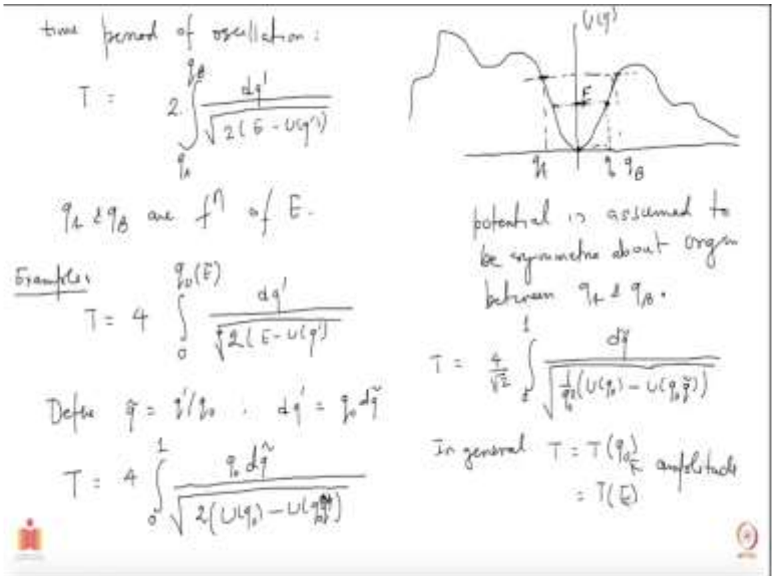

So, time period of oscillation which is just the time it takes from going from A to B and back that will be T this will be dq prime over this is what I wrote earlier 2 E minus U of q prime. And then you have to integrate from qA to qB and it will be the twice of it. Because it takes the same amount of time and as you can see that the values qA and qB are determined by your the total energy that the system has.

Because up to where this will climb will depend on the total energy that you give so whether it goes only up to here and here or up to here and here depends on the total energy that your system has. So, qA and qB are basically functions of energy of the total energy. Now, let us take a specific example maybe I should go to the next slide or here itself. So, I want to take this example.

My system is the following I am taking a one dimensional system described by the following potential. So, here I take the potential to be symmetric at least in this region, not necessary parabolic it is some function but it is symmetric. And beyond that it is not, so let us say it is like this and this one let say it goes whatever it does, but as far as let us say this point, this is let us lying on a horizontal axis between point A and point B.

So, qA, qB between these two points. It is a symmetric potential. So, that is one assumption. So let me write it down potential is assumed to be symmetric between qA and symmetric about origin between qA and qB that is the assumption. And let us say I start my system with some total energy E and let us say the energy corresponds to here and this point I call q naught that is the total energy of the system.

So, what will be the time period of oscillations when the system is put in this place, in this location? So, it has to go from here to there and back that will be the time period but you already know because I gave an exercise that the time it takes in going from here to there is same as it takes in coming back.

But then I have on the top of it chosen the potential to be symmetric which means the time it takes in going from here to here that is one fourth of the time period. So, here the time period T is 4 times the time it takes in going from 0 to q naught, I hope that is that point is clear the sole reason I why I took it symmetric around this was this symmetric that is nice. And ofcourse your q naught is determined by energy.

So, let me write q naught is a function of energy. Now what I want to do is the following let us define q tilde to be q prime over q naught. If I do so, then my dq prime is q naught dq tilde that is good. So, what is my time period now time period of oscillations? This is 4 times integral 0 to 1 because when your q prime is q0 your q tilde is 1. So, I have removed the dependence on energy from the limit of integral that is fine.

Now my dq prime is q0 dq tilde divide by 2. E is a constant. But E is equal to the value of the potential energy at this point. So, E is equal to U of q0. Let me write that U of q0 minus U of q prime and this entire thing sits under a square root that is good. Now I should not write a q prime I should write q0, q tilde, q0 and q tilde here not looking very nice, so I will clean it up q0, q tilde and this is in the square root.

All looks good here, okay fine. I can write this as T equal to 4 over square root of 2, 0 to 1 dq tilde over U of q0 minus U of q0, q tilde that is correct. And then you have 1 over q naught square, this entire thing in the square root. All I have done is put this q0 into the square root here that is all nothing beyond that.

So, note that in general this time period of oscillations will be function of q0 and this q0 is really the amplitude of oscillations that because that is the maximum distance it will come up to and which is determined by the energy as you said before. So, in general in general T is a function of q0 and remember q0 is the amplitude or I can also say that T is in general a function of E total energy.

Because q naught is determined by. Now let us ask whether it is possible to choose the potential energy to be such that the T does not depend on q naught and it is it is clear what we should do. If q if this entire function thing not a function of q naught then the denominator should not be a function of q naught.

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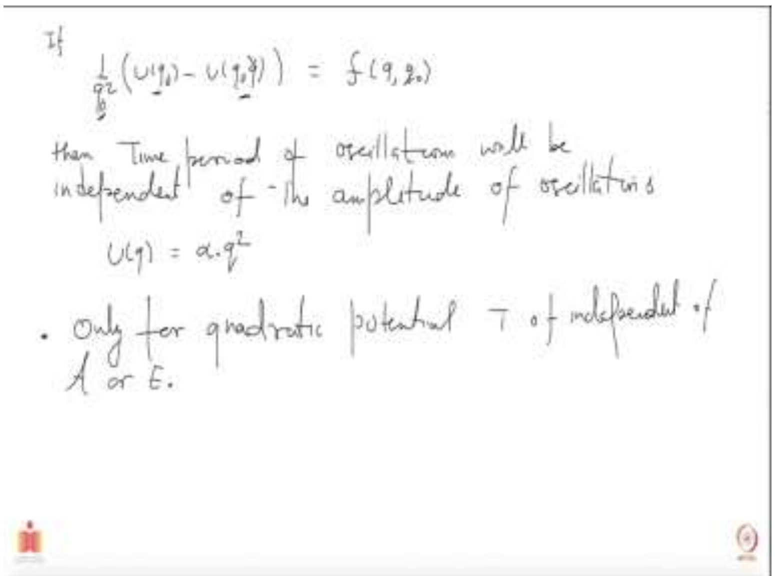

So, what we are asking is if we look at U q0 minus U of q0 q tilde 1 over q0 square that is what you had in the denominator. If this thing which is a function of q and q0. Let us call it f, if this were not dependent on q naught then the time period will be independent of q naught then time period of oscillations will be independent of the amplitude independent of the amplitude of oscillation.

Now, for what potential will does that happen? Clearly if I choose U of q0 or you have U of q let us write to be something some constant times q square then clearly you will have a 1 over q0 square from here and this will be proportional to q naught square. This will also be proportional to q naught square which will cancel with this q naught square and it will be independent of q0.

So, you see for quadratic potentials the time period of oscillations is in is independent of the amplitude and also of the energy of the system. If the potential energy of the system is not quadratic then this is not true. So, that is nice. So, let us write it down. Whatever if this then that so only for quadratic potentials T is independent of amplitude or energy.

That is good and this is the reason why when we study simple harmonic motion or small oscillations our time period was independent of amplitude or the energy of the system because we were under an approximation in which the potential energy was approximately to be quadratic. But the moment you deviate from quadratic approximation you will see that the time period will depend and that is what we will look at in a few examples, which I will take up next.