Introduction to Classical Mechanics
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Lecture 27
Forced Damped Oscillations

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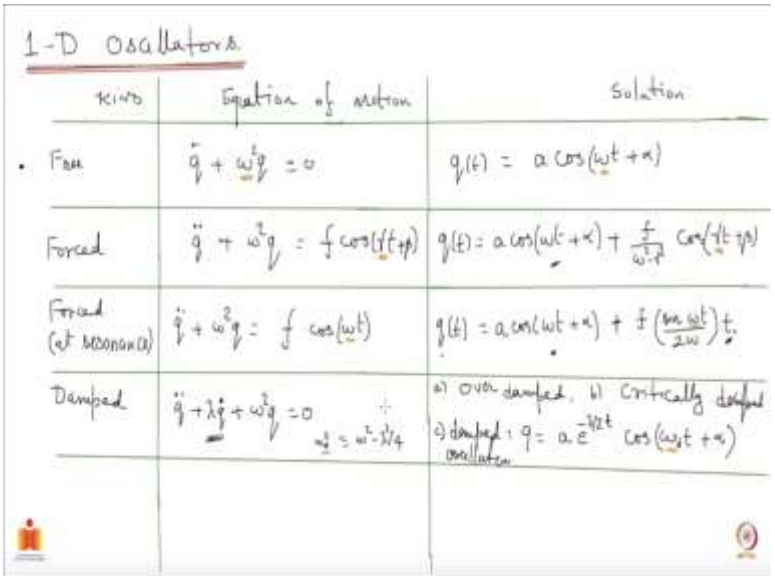

I have collected here all the results that we have derived in the context of one-dimensional oscillators. So, here you see the first entry is that of a free oscillator meaning there are no, there is no damping, neither, there are any external forces. So, here omega square is the natural frequency of the oscillator and this depends on the, for example mass or the spring constant, actually the ratio of it and the solution is a cos omega t plus alpha.

So, here the omega is same as what you have here on the left hand side and alpha and a are dependent on your initial conditions, what you choose the initial conditions to be and ofcourse you can choose your time such that alpha can be removed.

Then we talked about, the same oscillator, but now this time it is acted upon by some external agent and which is periodically and which is exerting a force which is periodic and that external agent has a periodic force with frequency gamma and beta is some phase that we have chosen for that oscillator and the general solution we found to be, you have the same thing corresponding to the homogeneous part.

So, if I put the f to be 0 you have free oscillator or equivalently a homogeneous equation. So, this is the homogenous part of the solution and then you have to search for a particular solution, which is what is in here. So, note that the amplitude of this piece is completely determined there is no freedom in choosing this.

So, f over omega square minus gamma square, there is no, you cannot choose some initial conditions and accordingly this will be different it is fixed for your system and also this part of oscillation is happening at the frequency gamma. Then we looked at forced oscillators at resonance.

So, here if omega is gamma this appears to be blowing up. So, we need to find the solution, so we wrote down the equation again and this time I have just chosen beta to be 0, there is no need to carry around a beta phase and we found the solution to be in this case given by ofcourse the homogeneous part will be the same as you got in here, it is the same identical here.

But now you see that the, this piece which is coming because of the forced oscillations, I mean because of the external force this is growing with time linearly. So, clearly after a while, the queue will not be small anymore and our physical assumption in writing down this will not anymore hold. The equation is fine, there is, it is mathematically, it is good, but our physical assumptions of small oscillations will break down and this will not be a correct description of the system.

Then we looked at damped oscillations, oscillators. So, we assume that there is a friction present, which can be described by this term. So, that the forces are linear in velocities. So, this is what you get, this equation is still a homogeneous equation and we saw that there are 3 possibilities: one is an over-damped case another is critically damped and in these both cases, we saw that the amplitude decays exponentially with time and then there was another case which we call damned oscillations. Maybe I should write here oscillations, where the solution has this form.

So, this is again not a periodic motion, but it is definitely oscillatory and the oscillations are having a frequency omega d which is lesser than the natural frequency because lambda is positive here, so you see omega d is lesser than the natural frequency and ofcourse the amplitude of these oscillations also decay with time.

So, if you wait long enough because of this exponential the q would be 0 or practically 0. So, that is what happens in damped oscillations. Now, what we want to do is, to look at again a one dimensional system, which is experiencing damping due to some friction which is present maybe air or whatever and also there is an external agent, which is applying a periodic force on it. So, that is what we want to do next and clearly. Let us see, what comes out of it. Let me not say beforehand.

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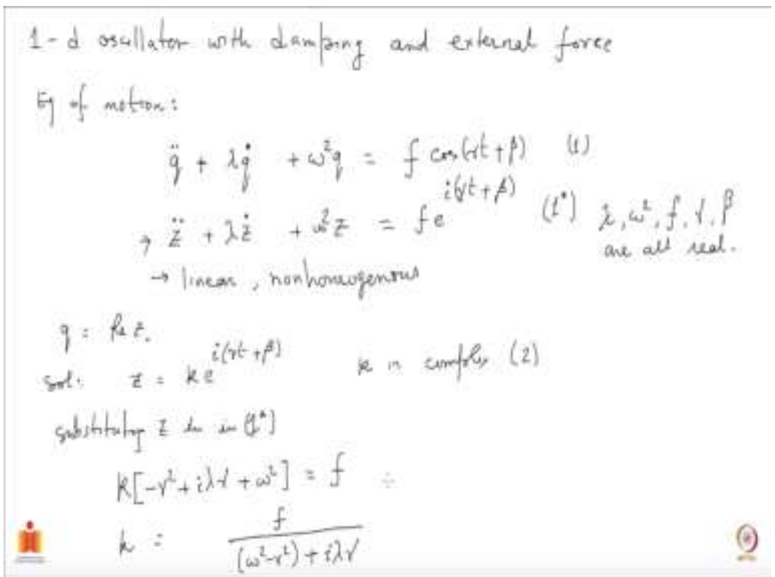

So, now we will look at one dimensional oscillator (what happened). So we will look at 1-d oscillator in, I mean with damping and external forces, external apply, external force. So let us, write down what the equation of motion would be. So, if the oscillator was not having any damping and without damping if its natural frequency is omega, then this would be the equation, this would be 0. So, let me not write 0 right now.

Now if you allow for damping of the kind which we have already talked then I see that I have to introduce a term proportional to velocity and then I allow for external force, which is periodic and I choose it to be cos gamma t plus beta. So, let me keep beta here, okay that is fine.

Now our experience with the case of damned oscillator, when we looked at this with right-hand side being 0, with that experience we can immediately see that we should look at the associated complex equation. So, instead let, instead let us look at the following equation. I will look at z double dot plus lambda z dot plus omega square z equal to f e to the i gamma t plus beta, where your lambda omega square f gamma and beta are all real numbers.

So, if I take a real part of this entire equation, I will be able to get this one, okay that is good. Now, this is again a linear equation, linear differential equation, but now its non-homogeneous, linear equation and non-homogeneous and our q, the solution would be real part of z and let me first, that is fine, that is okay, no problem. Now, let us me, let me look at the solutions of this equation, this complex equation.

So I, what I do is, I say that z is of the form k e to the i gamma t plus beta and we substitute this, because this is the only possibility that is going to work, because whatever z you take of e to the i exponential of something, they will all return back the z, but for this to work, your z should have the exponential this thing, otherwise, it will not work, so that is why we have this choice. But k is k is to be determined and k is complex. We are looking at complex equations. So, now you substitute this z in, in this equation and you get the following.

So, substituting z in let us call this as 1 star 2. Let us put a z the 1 in 2 in 1 star. So, what do you get? I hope you have, meanwhile I was writing you were deriving it. So, you get minus gamma square plus i lambda gamma plus omega square and this entire thing is equal to f. Now, you can write k as maybe yes here itself, so you can write k is equal to f omega square minus gamma square plus i lambda gamma.

I hope you have already noticed that the problem which you were having at resonance earlier is not here, is not blowing up. It is getting regulated, you will see more clearly when I write down the k in the polar form.

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k in polar form:

$$k = re^{i\phi}$$

$$r = \sqrt{k \cdot k^*} = \frac{f}{\sqrt{(\omega^2-\gamma^2)+\lambda^2\gamma^2}}$$

$$\tan\phi = \frac{\lambda\gamma}{\gamma^2-\omega^2} \cdot \cdot$$

$$z = \frac{f}{\sqrt{(\omega^2-\gamma^2)+\lambda^2\gamma^2}} \cdot e^{i(\gamma t+\beta+\phi)}$$

$$q = \frac{f}{\sqrt{(\omega^2-\gamma^2)+\lambda^2\gamma^2}} \cos(\gamma t+\beta+\phi) + a e^{-\frac{\lambda}{2}t} \cos(\omega_d t+\alpha)$$

steady state

transient

1-d oscillator with damping and external force

Eq of motion:

$$\ddot{q} + \lambda\dot{q} + \omega^2 q = f\cos(\gamma t+\beta) \quad (1)$$

$$\rightarrow \ddot{z} + \lambda\dot{z} + \omega^2 z = f e^{i(\gamma t+\beta)} \quad (1^*)$$

$\lambda, \omega^2, f, \gamma, \beta$ are all real.

→ linear, nonhomogeneous

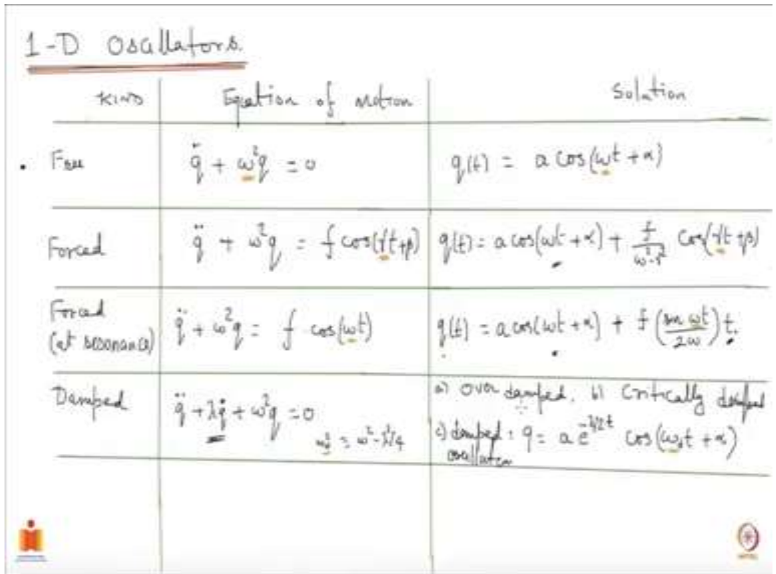
$q = \mathrm{Re}\, z.$

sol: $z = k e^{i(\gamma t+\beta)}$ k is complex (2)

substituting z in in (1^*)

$$k[-\gamma^2 + i\lambda\gamma + \omega^2] = f$$

$$k = \frac{f}{(\omega^2-\gamma^2)+i\lambda\gamma}$$



So, now what I will do is, I will write k in polar form. So, I am expecting k to be something like r e to the i phi where r and phi will be real and I have to determine r and phi and clearly how do you get r? r is just the square root of k-k star, this is complex conjugation. Now if you, make the substitution you are going to get the following, f over omega square minus gamma square plus lambda square gamma square the entire thing in the square root.

And what will be the phase phi? So, if you take 10 phi you will get lambda gamma over gamma square minus omega square, okay that is good. So, where were we? Real part of z that is correct and this fine. So, here I have to put in the k in the polar form and combine with this. So, I have here e to the i Phi, so that Phi goes and sits in the exponent together with beta. So the phase gets shifted from beta to beta plus Phi and the amplitude would be what you have here for r.

So our, where is it z? Z, so z becomes f over omega square minus r square plus lambda square, sorry not r square that was gamma, gamma square plus lambda square gamma square. Then you have e to the i. Let us look back, gamma t plus beta and then as I said, you will have alpha also not alpha phi, I have been calling it Phi. So, our solution q is the real part of z which is again this thing bored of writing this again and again, anyway let me write it down and then you have cos of gamma t plus beta plus phi.

Now this is not the full solution, this is just the particular solution of this equation, this is just the particular solution. Now if I add to this, solution corresponding to the homogeneous part, homogeneous equation if I put the right hand side to be 0 that will also be a solution. So, let me be more precise and put this as this is not the final solution. This part is just the particular solution and I should add to this a cos of what? Let us go back. So if this is, right hand side is 0.

This is the equation and this is the equation which you had for the damping case, this homogeneous equation. So, I will get what you got there last time, all those 3 possibilities have to be considered. But let us say I, let me go back and write down here. So, damped case a e to the minus lambda over 2 t cos omega dt plus alpha, omega dt plus alpha.

Let us write down, omega dt plus some phase alpha, something here and it was e to the minus lambda over 2 t. Let us check, so that is what you will get, that is the solution and as you can see that as time progresses this piece will just die away, it will just be not there. So, this is called the transient part. It is there while the transition is happening from this solution to only this part.

So, when everything has settled down, this part has disappeared your oscillator is oscillating steadily in this part. We say this is in a steady state oscillation. This is a steady state part. So, you see the oscillator after a while is just oscillating with the frequency of the external agent gamma that is the one. It is not the omega which determines the frequency, it is the gamma which determines the frequencies, but also note that the oscillator is not oscillating, as cos gamma t plus beta.

So, it is not in phase with the external agency. There is an additional phase phi, it is out of phase and that phi is given here and I will encourage you to figure out 5-4 different cases and the amplitude of oscillations is here. It is not blowing up as it looked like earlier and it is getting regulated by the coefficient lambda here. It is regulated by lambda. So, if you put lambda equal to 0 you see that at resonance omega is gamma and it blows up, but its getting regulated now.

Okay that is fine, and also note that all the parameters that you could have chosen depending on your initial conditions are in here this and this. A particular solution does not contain those perimeter. So, there is nothing you can control here by initial conditions. Once you have chosen the external agent your system which is going to oscillate and the lambda is given, your solution is completely fixed, this part the steady, the state part, there is nothing to choose and fix those parts we have.

Those parts which can be chosen are or fixed are in present in the transient part which dies away, anyway. So, this is for our one-dimensional oscillators. Now next we should be looking at a system near its equilibrium and then look at small oscillations around that equilibrium when there is external agency present and there is also some damping present. So, that is what we will look at next.