Introduction to Classical Mechanics
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Lecture 26
Damped Oscillations

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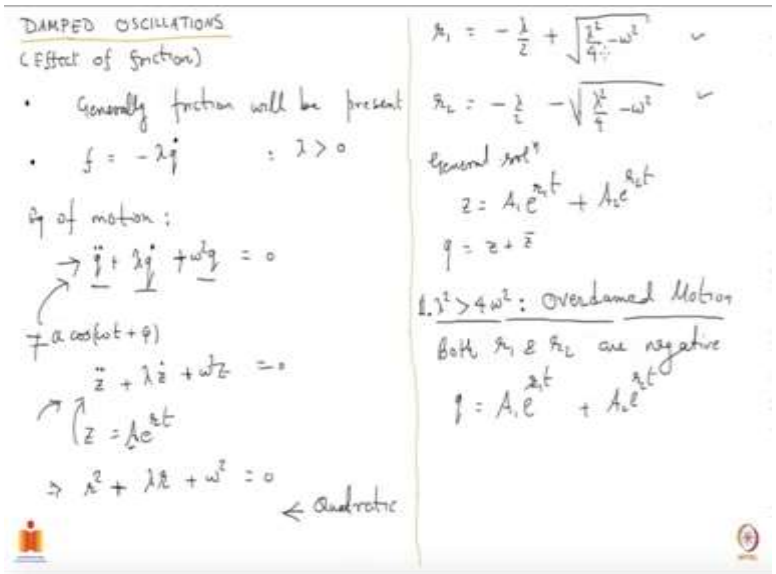

So, till now we have been looking at free oscillations and then last time we looked at forced oscillations. And we talked about resonance, and what the solution is at resonance. And we saw that the amplitude grows with time linearly. And the approximation of small oscillations will break down. We have not included any friction till now in our discussion. And clearly in any real system, you will have friction present. And that is what we want to take up today.

So, here we go. So, generally friction will be present, be present. That we know from our experience. And when our oscillations are small, typically the velocities will also be small. And usually the damping forces which are acting on the system, the frictional forces which are acting on the system, these forces will also be proportional to the velocities. So, in the limit of small velocities, the frictional forces which will act, or the generalized frictional forces which will act, they will, let us call them small f.

They will be of this form, proportional to the generalized velocity q dot, which we assume to be small. And then you have a constant lambda which is positive so that the force is opposite to this direction of motion of the system at that moment. Okay, and if I include this, then the equation of motion of the system, or the oscillator becomes the following. So you have, earlier you had q double dot, plus omega square q equal to 0, that is what we had originally, where omega square is the characteristic frequency of this system.

But now we are saying, we have an additional term which is lambda q dot. So if I put lambda equal to 0, meaning I remove all the friction, then you will just have q double dot plus omega square q equal to 0 and the system will oscillate at its natural frequency omega. And we want to know, now what happens if such a frictional force is also present. That is what we want to talk about today. So as last time, we are looking at a one dimensional system, so q is characterizing that one dimension.

And later we will take up multi-dimensional system and use these results to say something there. Now you cannot take the solution A cos omega t plus phi here, that will not work. If you put it in here, in this equation, it will not work. So this does not work. Because this is, let us say cos. I take double derivative, I get back cos again. But a single derivative will give me a sine. So this is not going to work. And a simple quick will make our life easy.

So instead of choosing q as the coordinate which is real, I go to complex variables and I say, I define a new equation z double dot plus lambda, where lambda is still the same as above real quantity. Lambda z dot plus omega square z equals to 0. Now, if I take the real part of this equation, then this is what I will get. So, real part of this complex equation will be my original equation. So if I can find solutions of this, I just take the real part of the solution.

Now, this one is good, because if I know that z equal to e to the r t will be a solution as far as the time dependence is concerned. And this will work because no matter how many times you differentiate z with respect to t, you still get the z back in addition to some factors which are not function of time t okay and that is wh, this trick is going to work. So let us substitute z equal to A, which is some constant e to the r t in the equation.

And this, when we substitute this in here, I get immediately r square, which is what you get by taking differentiation, doing differentiation twice plus lambda r, a single derivative will pull out an r plus omega square and A completely cancels out, because each term has an A. So this is what we get. And this is a quadratic equation which has two roots. Let me draw a line here. So as I said, this is a quadratic equation, quadratic. So I have two roots which I call r1 and r2.

And r1 is minus lambda over 2 plus lambda square over 4 minus omega square. And r2 is minus lambda over 2 minus lambda square over 4 minus omega square. These are the two roots we have. And in general the solution will be the following. Your z will be some constant A1 r1 t plus A2 r2 t. A1 and A2 are also complex. And to get my q, I should just take the real part which is same as taking z and z bar and adding them up.

Now there are 3 possibilities as far as the roots are concerned, these two. The argument of the square root could be positive, negative or 0, okay these are 3 possibilities. So let us look at these 3 possibilities in turn. So possibility number 1, which I will say, lambda square is greater than 4 omega square that is a possibility. And this is the possibility of, as you are about to see, it describes over damped motion.

Let us see why, why this possibility leads to over damped motion. So, this thing in here is positive. Now, as you can see that the r1 and r2 will always be negative. This one is negative, r2 is negative anyway, it is clear. You have something in the square root, two negative numbers added it is going to be negative. And you can also convince yourself that this is also, r1 is also negative. The, I mean the limit in which, the limiting value is 0. So when omega is really-really small let us say, you can drop omega in comparison to lambda square over 4.

In that case you just get minus lambda by 2 plus lambda by 2 that is 0. But if omega is anything but 0, this will be a small positive number lesser than minus lambda by 2. So, you will get a negative quantity. So in that case, let me write down here, for this case, both r1 and r2 are negative, as I mentioned just now. And clearly the solution q is an exponentially decaying solution, because you have e to the minus r1 t, some constant here.

Sorry, I should not have put a minus here that is what is confusing me. So q is A1 e to the r1 t, plus A2 e to the r2 t, and both of r1 and r2 are negative. So as time goes on, the amplitude decreases exponentially. So, clearly there is no periodic motion here. So, you start your system at some value of q, and as time progresses, it just damps out to 0. So, this is an over damped motion. Now, second possibility is that what you have in the square root vanishes. So, lambda square over 4 is omega square. Let us look at that possibility now.

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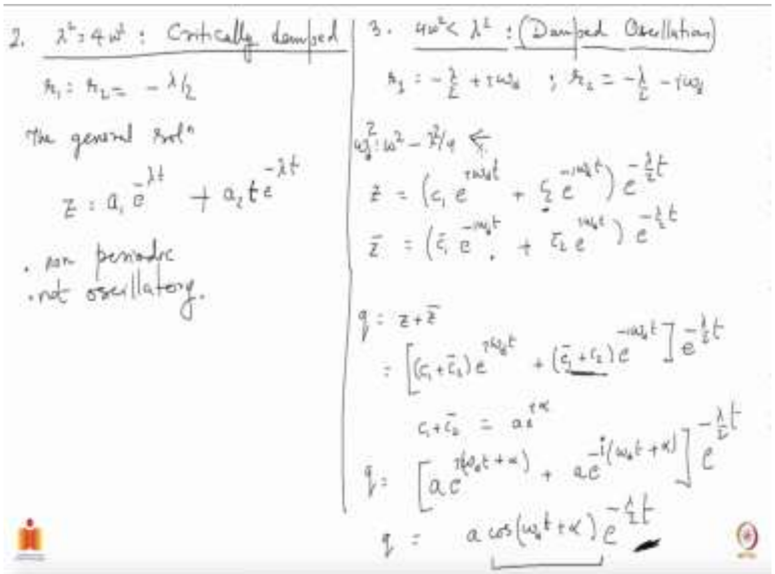

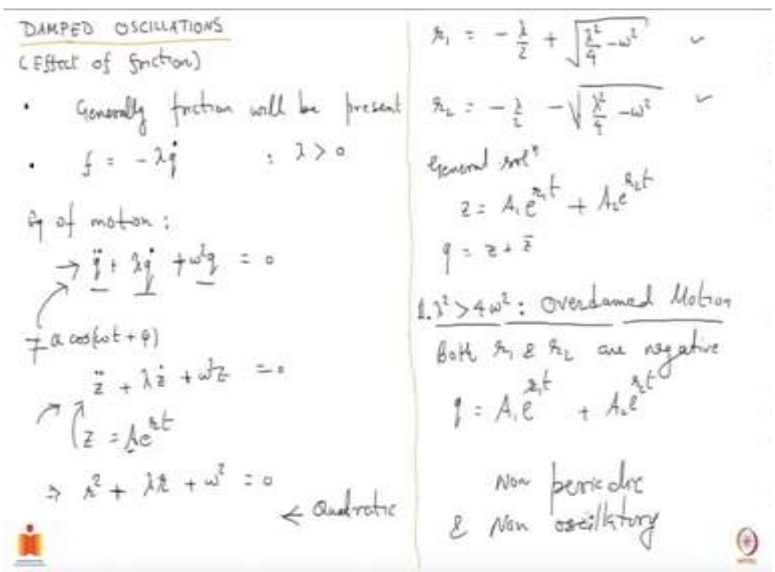

And possibility 2 is, lambda square is equal to 4 omega square and we will call it critically damped case. Let us see what happens in this case. So in this case, our r1 and r2 are both minus lambda 2 by minus lambda over 2. Let us write. So let us find the solution corresponding to r1. It will be just the general solution let me write. So, the general solution will be this, the linear sum of solutions corresponding to r1 and r2. And corresponding to r1, you get e to the minus lambda t, with some constant A1.

And because the roots r1 and r2 are degenerate here, they are repeated, you know that the second solution will be t times e to the minus lambda t. You get a factor of t here as you know from your study of differential equations. And ofcourse I should include another constant and add them up, to get the solution for my entire system. So, you can take real part of this and that will be your solution. But anyhow you see that this is again a non-oscillatory motion. It is not, I mean not, non-periodic motion.

Ofcourse it is not oscillatory also. But this is a non-periodic motion and also it is damped exponentially. Let me write here, in this case also. So, this first case was non-periodic, meaning the coordinate q never comes back to its original location, and it is also not oscillatory. Things are not oscillating as you will, as you see here. The same is true here also. This is also a non-periodic motion. And also this is not oscillatory.

Next let us look at the third possibility, which is, when the argument of the square root is a negative number. And that one we will call, so you have 4 omega square less than lambda square. And this we will call damped oscillations. As you can already see in the name, I am expecting that we will get oscillations unlike the previous two cases where there were no oscillations at all.

So here what we have is, your r1 and r2 if you see here, if I call this quantity as the negative of this quantity as omega d, I get 2 complex conjugates. So your r1 will be minus lambda over 2, and I am defining lambda square, omega square minus lambda square by 4 to be omega d. So what I am saying is, omega square minus lambda square by 4, I define this to be omega d. With that definition, what do I get? I get r1 to be minus lambda over 2.

Let me put a plus i, omega d. So plus i omega d and my r2 will be minus lambda over 2 minus i omega d. I should have put a square here. So these are the two complex conjugate roots and the solution is easy to construct now. You take the z to be c1, some constant i omega d, so I have put this piece, I will, e t t and minus lambda over 2t I will keep outside as a factor.

So let me first bring this one, c2 e to the minus i omega d t, and then I have e to the minus lambda over 2 t. So these are, this is the more general solution. Okay that is good. Now I can write down the z bar because that is what I need to add to get the real part. So z bar will be c1 bar minus i omega d t plus c2 bar e to the i omega d t, and this remains the same. And if I add these two up, I get q.

I am not worried about factors of 2, because that I can always absorb in the constants. So that is why I am writing z, q is z plus z bar. So if I add these two up, I get the following. So I take this piece and combine with this piece, because they have both e to the i omega d t. And I get c1, I get the following. I get c1 plus c2 bar e to the i omega dt, then you get these two pieces and this is c1 bar plus c2 e to the minus i omega d t, okay and this overall factor.

Note this factor is giving an exponential damping as time grows, this is exponentially decaying. Anyhow, so now I will define c1 plus c2 bar, these two constants, the sum of these two to be a e to the i alpha where a and alpha real. And then my q becomes e to the i omega dt plus alpha that is correct. And you have a a here, that is correct. And plus this is just a bar. And this is just complex conjugate of not a bar, this is just complex conjugate of this which means you will get, for this piece, you will get a e to the minus i alpha.

So, I get a e to the minus i omega dt plus alpha minus i alpha. So, I have taken out the minus sign that is why I get a plus in here, okay that is fine. Which is nothing but, so I can write down my solution q as a cos omega dt plus alpha e to the minus lambda over 2t. That is our most general solution. Here I am not again absorb all the overall factors of 2 in the constant a. So, this is the most general solution. Now, again this is not a periodic motion.

Meaning q never comes back to its original location, because of this, this factor. As time has passed, because of this the amplitude would have decreased. But this is in oscillatory motion because of the presence of this cos thing. So the pendulum or the oscillator is indeed oscillating about the equilibrium, but the amplitude is decaying with time. And this we have encountered also earlier. Also note that the oscillator is not oscillating with the frequency omega, which was the natural frequency of the oscillator, of the free oscillator let us say.

But it is oscillating with the new frequency which is omega d and where omega d is this quantity here. And clearly it is smaller than the omega, this minus lambda square by 4. Let me write these two things down.

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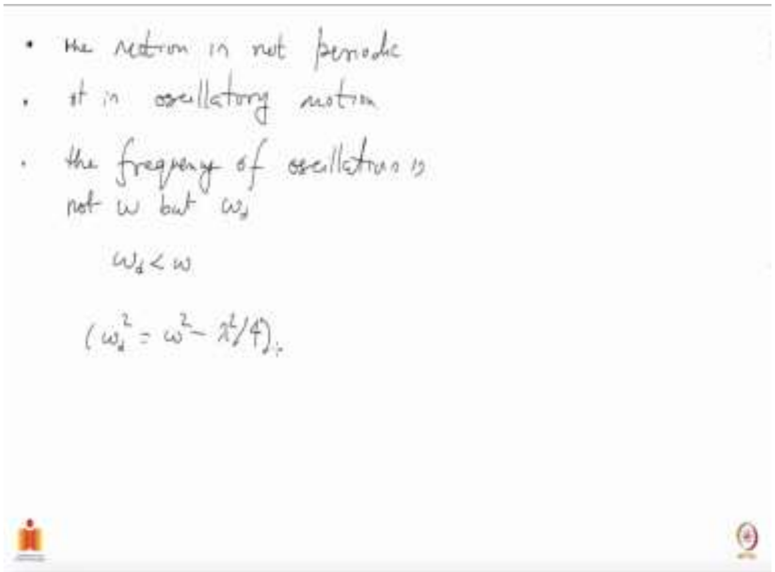

So one in this case, where I have damped oscillations, I do have oscillations, but it is not periodic. So the motion is not periodic. Oscillator never returns to its original location because of that exponential damping factor. But it is oscillatory, it is oscillatory motion. And also, the frequency of oscillations is not omega, but omega d.

And omega d is less than omega. Recall omega d square is omega square minus lambda square by 4, I believe. Lambda square by 4, that is correct. That is good. Next what we should do is, we should look up, look at a system, which is both experiencing a damping, a damping because of its environment, where it is for example, a pendulum could be oscillating and it will be experiencing frictional forces due to the air. And on the top of it, there is a external agent, which is applying a periodic force on it. So, we want to next take up damped system, a damped force system. That is what we will do next.