Introduction to Classical Mechanics
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Lecture 25
Forced Oscillations

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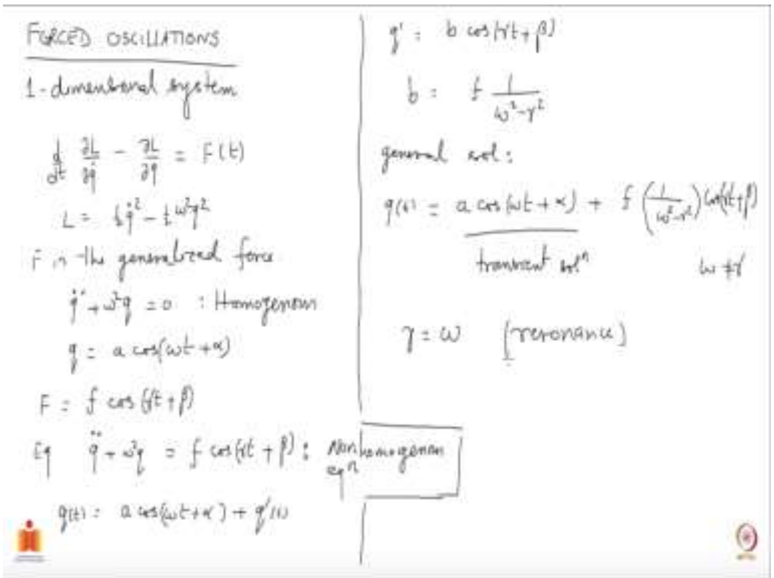

Till now we have always been talking about free oscillation of a system today we will start looking at forced oscillations and I am going to be interested in 1-dimensional system, 1-dimensional system. Will talk about system in more than 1-dimensional later, that is one thing and then I will be interested in looking at this 1 dimensional system and will assume that there is a force which is acting on the system externally.

And that external force is a sinusoidal force because that is what most generally you will be interested in, because if you have your system on which a periodic force is acting then you can decomposed it into sinusoidal forces, sinusoidal components and that is why we interested in looking at forces which are sinusoidal.

So, let us look at the equation of motion for such a system, so I will have Euler Lagrange equation will be d over dt, del L over del q dot minus del L over del q equals, here you will have short generalize forces remember our first two lectures where we talked out talked about generalize forces. So, that I put in here and the Lagrangian of my system will be described by half q dot square minus half omega square q square.

So, which is which means that the omega is the natural frequency of the system and I have chosen the coordinate such that the coefficients here is just half everything absorbed, everything else is been absorbed into q. Let me write down here f or not there maybe, F is the generalized force. Now, if f is equal to 0 it vanishes then you have a free system and the solution to the free case you already know is this q is a let me see whether I am using a yes, a cos omega t plus alpha.

And your equation of motion in this case is q double dot plus omega square q is equal to 0 and remember this is a homogeneous equation. Now I want to assume sinusoidal force aaas I said, so I will take the generalize force F to be some constant f small f and then you have cos gamma t plus beta. So, the frequency I take to be gamma of this external a and c.

And now our equation of motion is q double dot plus omega square q equal to f cos gamma t plus beta. Note I have taken different alpha and beta here, so I mean you can choose one of them to be 0, so let say you can choose alpha to be 0 which is basically equivalent to choosing what is the origin of time. I mean you can absorb that alpha in the definition of timeline you can choose your new time such that there is no alpha here.

So, this is what I meant, so you can redefine the origin of time and this alpha will go away but then here I keep it general, so that you have a some phase here corresponding to the free case. Anyhow so that is what it is and this equation is not homogenous, this is a non-homogeneous equation. And the most general solution is to this non-homogeneous equation q of t would be the homogeneous part which is a cos omega t plus alpha.

Plus, a particular solution let I should find out a particular solution to this non-homogenous equation and then I will find the then I will get the full solution. So, that is what we have to do. Now, you can let me draw vertical line here. So, let me search for solution a particular solution of this form so I say q prime will be some b cos gamma t plus beta.

So, that is what a look for and that I should do because that is what you have on the right-hand side. And if I substitute this in the equation of motion here in this one I immediately get b equal to f 1 over omega square minus gamma square. So, the most general solution I have is the following. I should add to the particular solution the solution corresponding to the homogenous part and this is what it is plus the particular solution.

And ofcourse, I have forgotten no nothing it is fine, so your where is b is this and then you have cos gamma t plus beta, so and where omega is not equal to gamma because that omega equal to gamma this solution blows up. So, now imagine, this is this part is called a transient solution because if imagine there were there was friction present. Let say the pendulum is experiencing some friction with will be the case in reality.

And then let say also imagine that there is no external force acting on it and you have a free pendulum you I mean a pendulum which is experiencing some friction after a while it will stop. In this solution will just go away after some time if there was friction present and that is why this is called transient.

So, that is the solution for forced oscillations without any friction let me write here (trans) but this is not valid when your gamma the frequency of external agency is same as the natural frequency of the system it looks like this solution blows up.

So, it is valid here, so will next look at the problem near on your resonance so when omega is same when gamma is same as omega we say we have resonance. And that is what we want to look at now. So, let me write it down.

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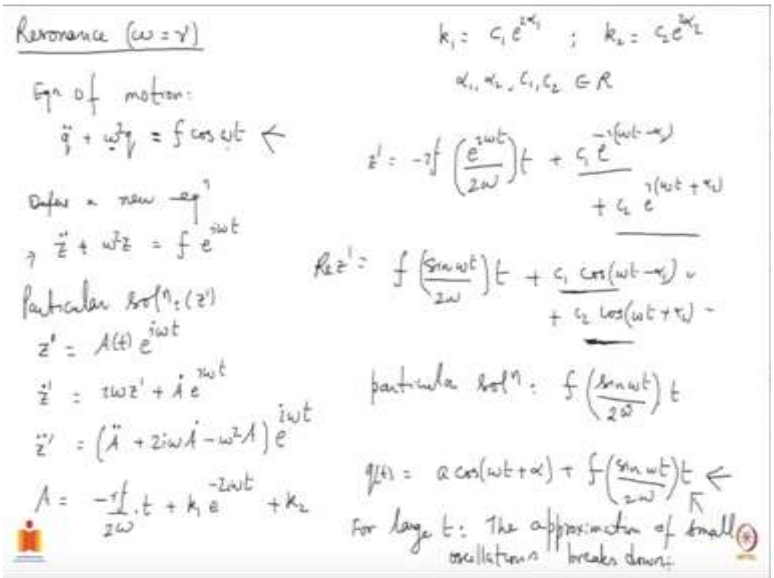

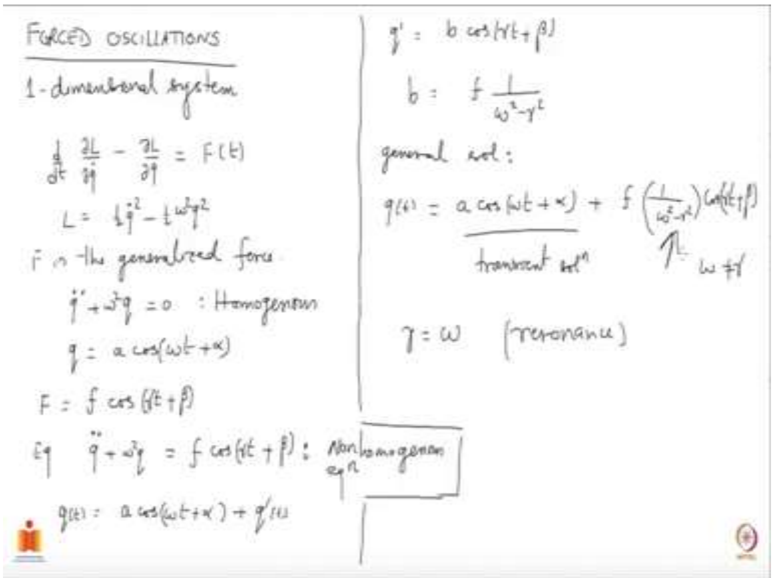

So now our equation of motion is the following, q double dot plus omega square q equals f cos omega t, I just put the face to 0. If you wish you can bring it back but I will just put it to 0. And this omega is same as this omega here this one and this one are same. What I can do is I can go to complex variables and instead of using q as instead of using the real q I use z.

So, let me define a new equation z double dot plus omega square z equals f e to the i omega t. And if I take a real part of this equation I will get back this one where I will define q to be the real part of the z. Now, as before because this is a non-homogenous equation I should search for particular solution and ofcourse I know what the solution to homogenous part is.

So, let us go ahead and search for a particular solution. Let us try the solution of this form so I say the particular solution I will denote by z prime. So, let us say the particular solution z prime is some function of t which is A of t times is the i omega t let me try this one out. You can check that z prime dot is i omega z prime plus A dot e to the i omega t. And if you take a second derivate of this you are going to get A double dot plus 2i omega A dot minus omega square A should take this I hope I am not making any mistakes, e to the e, e to the i omega t.

Now, you can substitute all the 3 here one z prime, z dot prime and z double dot prime put them in the equation of motion here and obtain the following. If you do so you get the following you get A to b minus i f over 2 omega times t plus k1 e to the minus 2 i omega t plus another constant k2. So, k1 and k2 are complex constants, that is good.

Now, if I have found my A I have my Z prime which is the particular solution. Now what I will do is I will write down k1 as C1 e to the alpha 1 and k2 as C2 e to the i alpha 2, where C1, C2 alpha 1 and alpha 2 they are all real numbers, let me write down alpha 1, alpha 2, C1, C2 they all belong to real numbers and you get the particular solution to be now z prime equals to minus i f e to the i omega t over 2 omega this entire thing is multiplied by t plus C1 e to the minus i omega t minus alpha 1 plus C2 e to the i omega t plus alpha 2.

That is your particular solutions, so if I take real prime of z prime, sorry real part of z prime then that is what will be the particular solution of this equation. Now, look at these two parts, these two are let us take the real part of z prime. Real part of z prime is f sine omega t over 2 omega t plus C1 cos omega t minus alpha 1. And C2 cos of omega t plus alpha 2.

Now you see these 2 terms, this one and this one these are just cos omega t and something some phase and these are the solutions to the homogenous equation. The homogenous equation associated with this non-homogeneous equation. So, these two you can drop, you can put the constant C1 and C2 to 0, and you can do so because these two because this thing is anyway available in the homogenous equation.

But if you do not want it does not matter you can keep them and when you add everything you can combine this, this and the one with the homogenous again to one function which will be cos omega t plus some phase times some constant, some constant amplitude. So, either way whichever way you prefer you can drop, you will have only this peace and the homogenous part.

So, I will put the constant C1 and C2 to 0, so my particular solution is f sine omega t over 2 omega times t, and as I said I can write down the my solution q of t to be some constant a cos of omega t plus alpha plus the particular solution. Now, you see I do not have any singularity at omega I mean there is no nothing singular going because omega is equal to gamma there is no singularity here. As we had in this case, if you see here.

Okay, that is good so this is the write we are doing it now if you look at the solution carefully and pay attention to this term, you see it is ofcourse there is a harmonic variation sine omega t here but you also have a t proportionality here. Which means that your solution your displacements q of t they are growing linearly with time, so this grows linearly with time.

So, at some point of time the displacement will be large because as time grows the displacements grows linearly. And your approximation that you are close to the equilibrium will no longer whole true and all the other terms that we have dropped in linearizing the equations of motion. They would play a role and the solution will not be valid but if you are looking at only small values of t then this is fine, then this is the solution.

So, for large t the approximation of small oscillations breaks down, next we should look at dammed oscillations and then we will have a both dammed as well as forced or everything present together and will look at the solution in that case and eventually we would like to see what we can say when we have not just a system of 1 degree of freedom but as we have been discussing before what happens if I have a system of more than 1 degrees of freedom can I do again what I was doing and bring in normal coordinates. So, that will be the, our next task.