Introduction to Classical Mechanics

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Lecture 24

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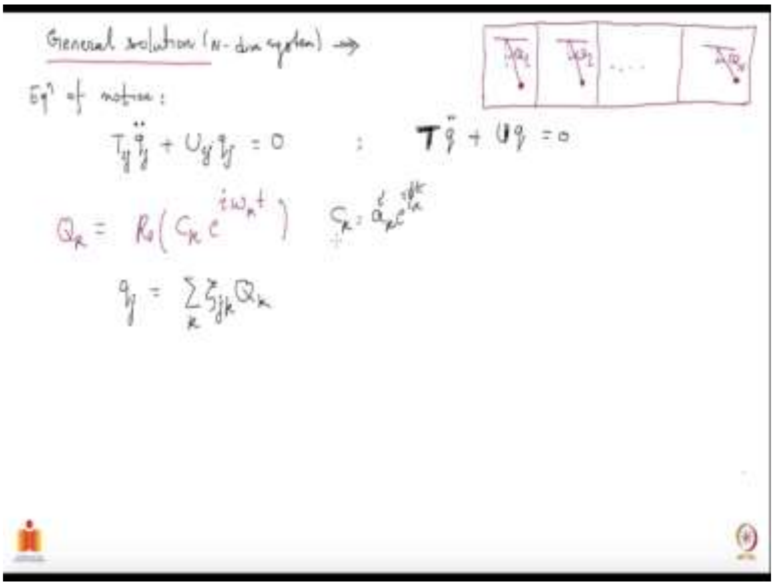

So, based on the discussion that we have had till now, we are ready to write down the general solution for and N dimensional system which is undergoing some small oscillations near its equilibrium configuration. So, for N dimensional system, so if you take the Lagrangian and from that you write down the equations of motion this is what you are going to get.

So, our equation of motion would be Tij that is your kinetic energy matrix qj double dot ofcourse you have second derivate and then you have the potential energy term Uij qj and this is equal to 0. I can write the same equation in matrix form so the same thing looks like this let me write it down on the right-hand side.

So, let me denote by this T the matrix corresponding to this elements Tij and then we have q double dot q is a column vector plus U it is a matrix again and then you have the column vector q and this should be 0. And what we want to do is, we want to write down the general solution for q. Now, this is not difficult now because we have already done most of the work in fact we have done all the work all we have to do is just write it down.

So, what we will do is we will recall that our system behaves as if it is a collection of N independent harmonic oscillators, I am assuming all the frequencies to be non-zero here, all the oscillators have non-zero frequencies. So, the picture is the following. So, the system this system now if you choose the normal coordinates then it looks like the following maybe I should use some, why it does not work? Yes okay.

These pictures will they are not necessary but they do help in remembering certain aspects so yes here, so you have your system appearing as if there are I mean under the right choice of coordinates it appears as if you have several harmonic oscillators, one dimensional harmonic oscillators. And each of them will have a different frequency and I am assuming that they are non-zero right now.

And this will be the coordinate q, remember this is just a cartoon, so q and so and oscillators and these qs will give you the coordinate which characterizes the displacements. Okay very nice so, now it is clear what the general solution would be, it would be just a linear sum of all these normal coordinates.

Now, what do you have at your hand when you disturb the system? So you can think of your disturbing your system in such a way that you choose to oscillate the oscillator capital Q1 such that it has a phase phi 1 at time T equal to 0 and some amplitude. So, it will have its own amplitude and some phase. This one have will have its own amplitude and some phase and so forth.

So, those are the things which you control, they depend on the initial conditions, because given one dimensional oscillator that is what the freedom you have in choosing. Which means the following that the kth normal coordinate, I can write as Ck e to the I omega kt and I should take the real part of it. Ck is complex which includes the amplitude and the so let us say I write Ck not here.

So, the phase part of Ck combines here and gives you the phase of the oscillator k and the amplitude here the radial part gives the amplitude, so that is what your normal coordinate k is okay and we already have seen that the transition from your q, small q these coordinates to the normal coordinates is through a linear transformation.

Which means I will write down, let me use Q, I should use black. So, my qj the displacement qj will be the following, it will be a linear sum of all the one dimensional oscillators and they will all come with some coefficients zi jk, so there is a summation over k implied here may be I can make it explicit.

And these zi will be determined by… we have to calculate, we do not have a control over what zi is there should be clear. See what you can control is what is Ck and Ck is two quantities basically the, so if I write Ck as some ak e to the power i phi k. So, you can control what phi k is, what ak is the amplitude and this.

But beyond that for one dimensional oscillation there is nothing for you to choose and if you can reduce your system to N oscillators each of them one dimensional you correspondingly choose the phi k and ak for each of them and beyond that you have no freedom left to choose anything, which means that this zi is completely determine by the problem itself there is nothing for you to choose. This is the point I wanted to make. I will remove this now okay good, yes.

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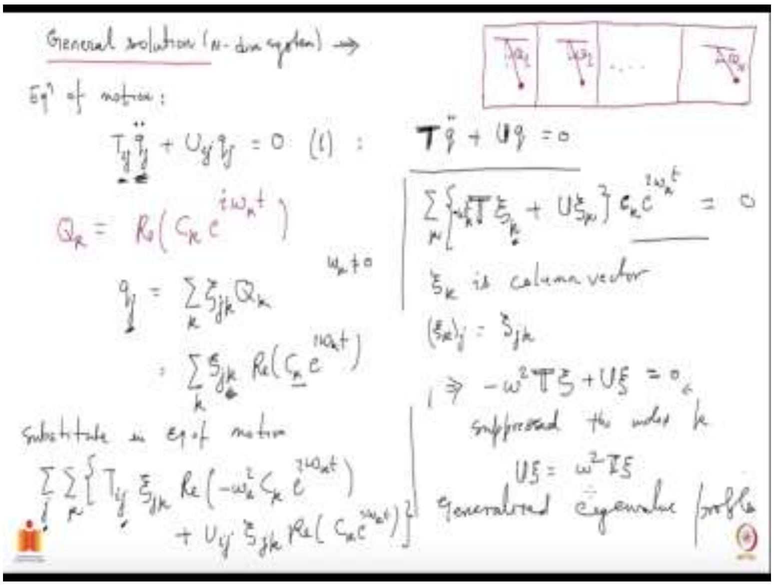

So which is basically I can write this as Zi jk times real part of may be in the next line Zi jk times real part of Ck e to the i omega kt. Note that your Zi jk all these elements are also real they have to be real because here qj is a real and this part is real so this has to be real otherwise your displacements would not be real.

And I have already made the assumption that I am taking all the omegas to be non-zero. Now, I take this qj and substitute in my equation of motion. So, substitute in the equation of motion and what do I get I get the following. So, you already have a summation over j in this implied here which I have not made it explicit but now I will.

So, I had summation over j and then I have a summation over k then your Tij here Tij then you have your I have to substitute Tj I had take the second derivate which brings the i omega 2 times which gives a minus omega k square. So, I get Zi jk then your real part of minus omega k square then you have a ck here then you have e to the i omega kt.

And then you have your potential energy term which gives you the following Uij may be the I can write down in the next line plus Uij xi jk and then real part of Ck e to the i omega kt, so that is what you have. We can collect certain terms here and write it in the following fashion. So, from here so now I have summation over k and I am going to use matrix notation, so I am going to write the summation over j using matrix notation.

So, this tij is our matrix T and Zi jk, so it is basically Zi is getting dotted into the T that is the product let me write down and then maybe it will be easier to see what I am trying to say here. So, minus omega k square T Zi of k I will tell you what Zi of k is in a moment plus U Zi of k okay, okay fine, yes I should have guess why no problem, e to the sorry this is c, ck e to the i omega kt equal to 0.

So, let me tell you what Zi k is, your Zi k is a column vector you see Zi k, this Zi k does not carry two indices does not carry jk, so it only one so it is a different quantity and what I have done is a have Zi k as a column vector, c o l u m n vector who's jth element, so if you look at this vector and look at its jth element this is Zi jk. Okay that is the definition of this column vector Zi, Zi k.

And if you see this result this expression, we have sum over k and on the right-hand side you have 0 and each term corresponding to each k comes with the coefficients ck which you control and everything here in these curly brackets is not controlled by you. So, if this entire thing has to sum up to 0 it better be that the term in the curly brackets vanish.

Otherwise it is not possible to make it 0 for whatever Ck you choose because Cks are in your hands, so for that to happen this quantity in curly bracket should vanish and let me write it down. So, this implies that your minus omega square I am just dropping the k for moment not for… I am just dropping it. So, the following should happen.

So, T Zi plus U Zi should be clue to 0 because your ck are arbitrary, determined by the initial conditions. So, that is what the Zi should satisfy as I was saying earlier here that this are determined by the problem the Zi k and that is what you see now that the Zi k are determined by this equation I have just suppressed the index and you will see immediately why I have done so, okay and this equation which you have here let me write it slightly differently you can write is as U of Zi as omega square T Zi, this is called the generalized Eigenvalue problem.

The normal Eigen value problem would be like U of Zi is equal to some constant times Zi this is the normal Eigenvalue problem but this is the this is what is called generalized Eigen value problem instead of having your right hand side being proportional to Zi you have a constant and again a matrix time the Zi. This is called generalized Eigenvalue problem.

Now, you see if I give you system and ask what are its characteristic frequencies what are the frequencies of the normal modes you do not have to first put the Lagrangian in as a sum of squares which the way I was telling you earlier. You can simply do the following, you look at this equation and from here we can conclude how to find out the Eigen frequencies.

And simple because these are set of liner equations let me write it down in the following form.

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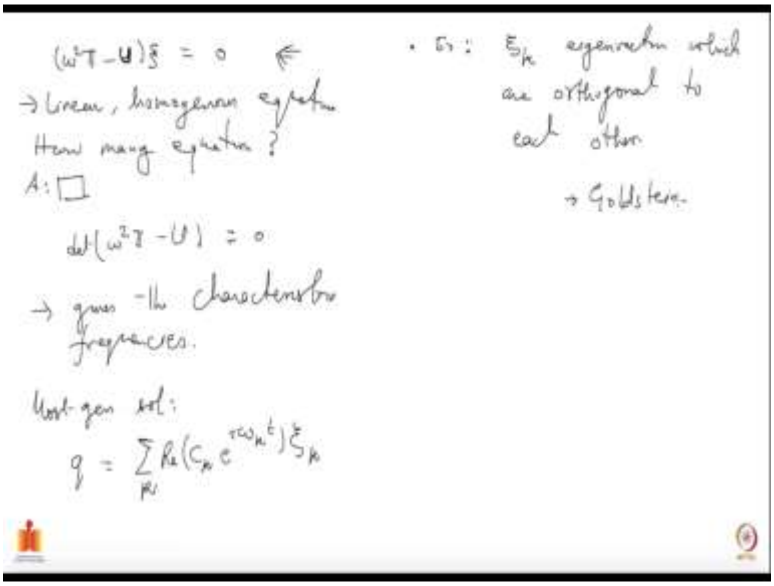

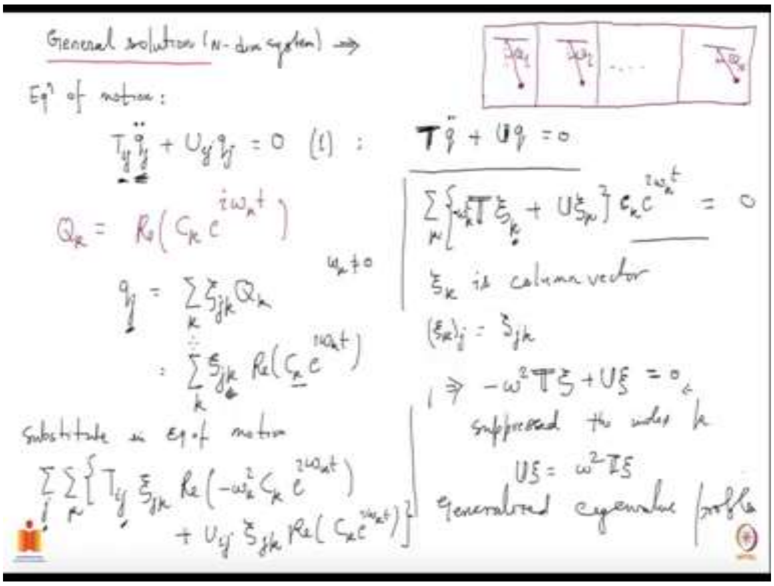

I write it as omega square T minus U, Zi equal to 0. These are linear homogenous equations, how many of them are there? These are how many questions? Please supply the answer here. Now, if this set of equation has to have a non-trivial solution, meaning a non-trivial Zi which will make this happen then it is necessary that we should have determinant of omega square T minus U equal to 0.

So, if we can solve this equation which is a polynomial equation of degree what you find here. And then you can solve the polynomial equation and get the characteristic frequencies, so this will give you okay it may also happen that some of your omega square turn out to be 0, but that is fine. Now, let say I want to write down the most general solution to our problem where the frequencies are non-zero then it would be just this, this qj.

So, your most general solution is q let me write down the solution in the matrix notation so q is a column vector here. So, you had a qj Zi k jk now this will become a column vector Zi of j okay. And these are anyway here. So, your most general solution is the following. And let me make the summation over k explicit, also I would request you to check that this problem if you are looking at this generalized Eigen value problem.

The Zi of k in our case is a set of Eigen vectors which are which are orthogonal, orthogonal to each other ok you will be able to do this based on the discussion which we have had in the last couple of hours or you can also refer to some book like Goldstein, okay so we have written down the general solution and will continue with more discussion on harmonic oscillations or small oscillations in the next video.