# Introduction to Classical Mechanics
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**Lecture 23**
**Coupled pendulums, Beats**

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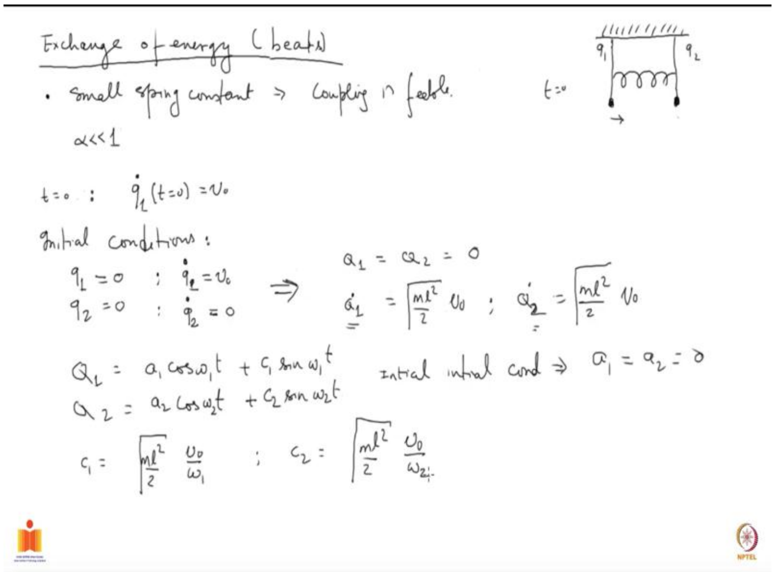


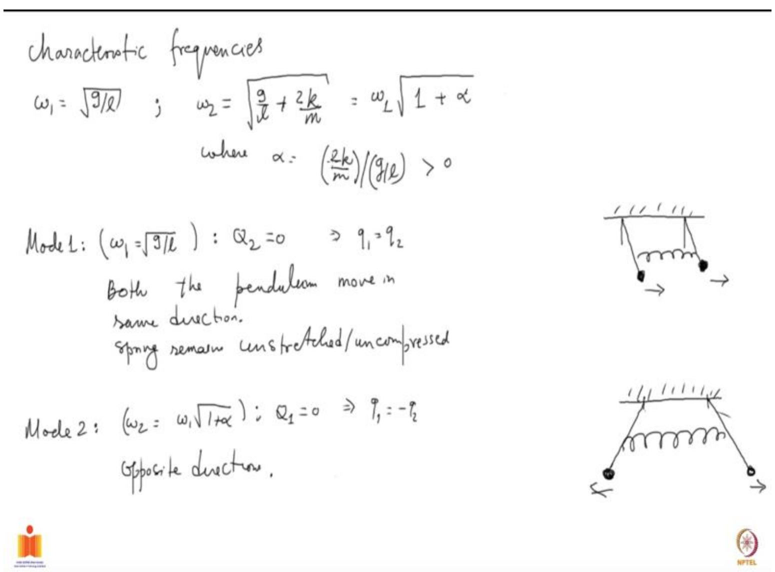


So, here I have redrawn the setup and by showing this spring I am assuming a very small coupling. And what we are going to see is that there will be an exchange of energy between the two. I will exchange of energy between the two pendulums and this is the phenomena of beats. Let me explain to you what is going to happen. So as I said so what we will see is that let us say at time t equal to 0 you gently move it this way. Now because the coupling is very small it will not immediately push it away, so it will you will see what is starts happening today.

It will, you see will that it will start moving slowly a little bit and something very interesting will happen eventually, so that is what we want to see. So first point is that spring constant is small which ensures that the coupling between the two pendulum is also very feeble. Which amounts to saying that my alpha is much smaller than one the alpha which I define here, it is here, this counted, because this is control by k.

Anyway, so as I said lets both of them be hanging down and then at some point which we call t equal to 0, we will give first one let say this was the q1 which you were saying and this is the q2. So, at this time we give q1 a gentle velocity, so q1 at t equal to 0 let us call it v0. So this is what we do, so our conditions, initial conditions at time t equal to 0 are the following. So, you have both the pendulums at rest so even if you give a gentle nudge at time t equal to 0 it will still be at 0. q2 is also equal to 0 and if you look at the velocities your q1 dot is v0 so we have given it a velocity. We have not done anything to the second one so at time 2 equal to 0 this will still be a rest.

And this translates to the following for the normal coordinates, so the same initial conditions if you look at look for the normal coordinates they translate to the following. So, this you can check immediately that at time t equal to 0 this is all t equal to 0 whatever I am writing here. So Q1 will be same as Q2 and it will be 0, just because the small q and small q2 are 0 and capital Q1 and capital Q2 are linear combination, so they will also be 0 and you can also immediately see that Q1 dot will be ml square over 2 and the square root times v naught.

And Q2 dot will be ml square over 2 in the under root times v naught. Now we can solve for small q and small q2 easily because we know that the capital Q1 and capital Q2 the normal modes are anyway harmonic oscillators. So clearly that Q1 is, the most general solution is a1 cos omega 1 t plus c1 sin omega 1 t that is the most general solution. And similarly, for Q2 you have a2 cos omega 2 t plus c2 sin omega 2 t.

And, now if I put my initial conditions then you will let me just write initial conditions they imply that a1 and a2 are 0, why? Because see Q1 and Q2 are 0 at time t equal to 0, so clearly when I put t equal 0 these two should vanish. Otherwise, I mean this will vanish but this will give non-zero contribution unless a1 and a2 are 0. So clearly, the initial conditions imply that a1 is same as a2 and they are both 0. That is one thing, now we still need to determine c1 and c2 which you can get by using these two at the initial time which are at the initial time. And check that you get the following, check that c1 turns out to be ml square over 2 square root v naught over omega 1 and c2 is ml square over 2 v naught over omega 2, and this will be easy to verify.

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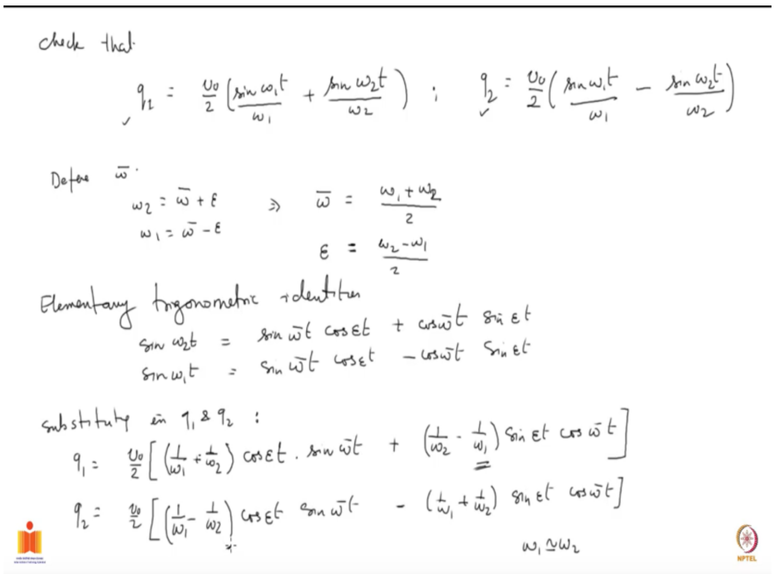


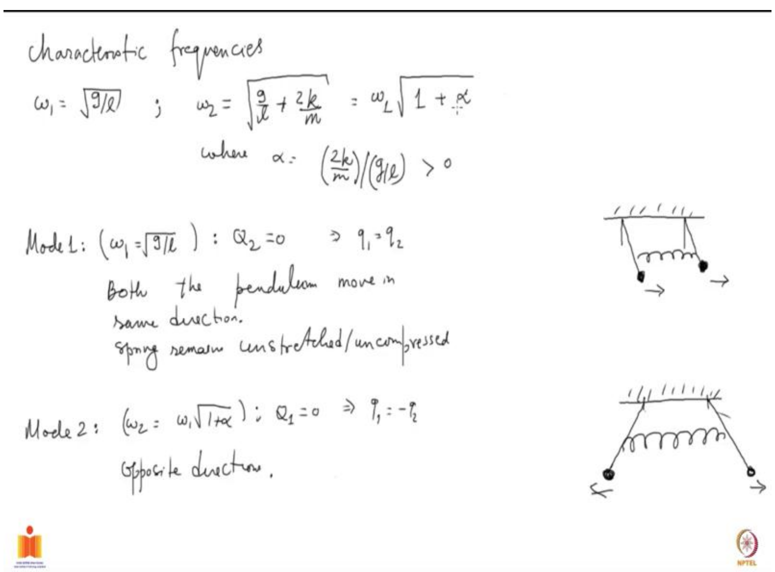


Also, you can immediately check that if you substitute all these constants which you have determined just now in the definition of q1 and q2 not definition, the expression of q1 and q2 in terms of capital Q’s. The normal coordinates you are going to get the, so check that q1 is v naught over 2 sin omega 1 t over omega 1 plus sin omega 2 t over omega 2. And, for q2 we get again similar looking thing but with the minus sign, sin omega 1 t over omega 1 minus sin omega 2 t over omega 2.

That is easy so please check that, now what we will do is we will define something call omega bar so we will define, omega bar to be the following. So, what I am going to do is I am going to write omega 2 as omega bar plus epsilon and omega 1 as omega bar minus epsilon. And, of course you can find out what omega bar and epsilon are because you have two equations and two unknowns which is, which will give you that omega bar is just the average of omega 1 and omega 2. And, epsilon is the difference of the two frequencies, you can see why I am putting epsilon, epsilon usually denotes small quantities.

The reason being because I have taken the coupling to be small, where is it, if the coupling is small then omega 2 is also close to omega 1 and that is what I am doing. So I am just defining omega bar which is the average of the two which will be almost omega 1 and epsilon will be close to 0 so that is why I am doing. Now you can use some elementary trigonometric identities, which I will write here. So, elementary trigonometric identities, you can check that this is what you get these are, so what you have to use a sin omega 2 t this you can write as sin omega bar t cos of epsilon t plus cos of omega bar t sin of epsilon t.

And, then you have sin of omega 1 t which you can again the same expression cos of epsilon t minus cos of omega bar t sin of epsilon t, that you can check easily. And you substitute in q1 and q 2 these things, so here you go to here and substitute what your sin omega 1 t is from here and what your sin omega 2 t is from here and check that you get the following. Substituting in q1 and q2 we get the following we get q1 as v naught over 2, 1 over omega 1 plus 1 over omega 2 cos epsilon t times sin omega bar t.

Then you have one more term, which is 1 over omega 2 minus 1 over omega 1 that is good, sin epsilon t cos omega bar t that is what you get for q1. And, let me write for q2, q2 you get the same thing apart from small differences, let me see sin, cos omega, so you get this 1 over omega 1 minus 1 over omega 2. I hope all the signs I have derived correctly but please check them, cos epsilon t sin of omega bar t, that is good, minus 1 over omega 1 plus 1 over omega 2 sin epsilon t cos omega bar t, that is what you can check.

Now what I will do is I will put omega 1 to be close to omega 2 which is the case, let us say these are two very close. Now if that is a case then this term the overall coefficient is almost 0 and similarly here so they will drop out.

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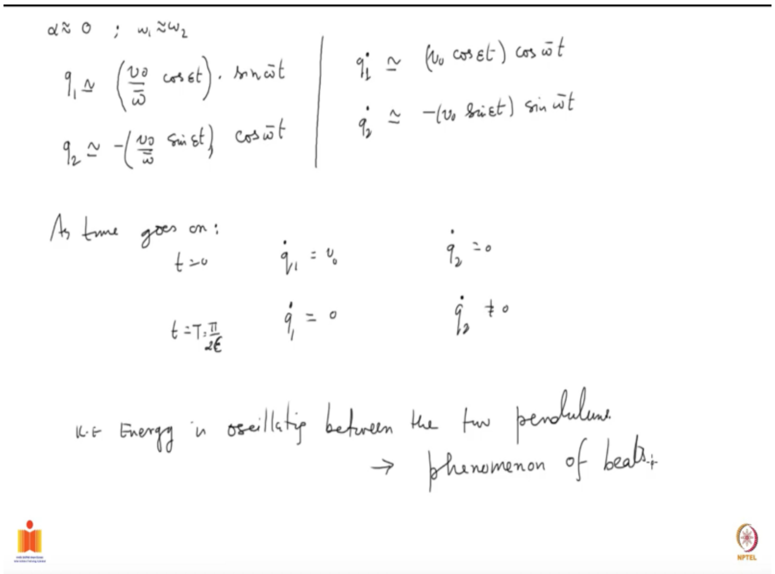


And in that limit of small coupling meaning alpha is close to 0 or which is same as omega 1 is same close to omega 2. You get the q1 to be approximately v naught over omega bar cos epsilon t times sin of omega bar t and q2 will be almost minus v naught over omega bar sin epsilon t and cos omega bar t, that looks good.

Let us check whether it is looking reasonable so let us say we put time t equal to 0 in this and clearly both of the pendulum are at rest because this one become 0 here and this one will become 0 here, so they are both at rest. Now let us find out the q1 dots and q2 dots, the velocities at any time please check that this is what you get v naught cos epsilon t times cos of omega bar t. And q2 dot you get minus v naught sin of epsilon t sin omega bar t.

Let us put t equal to 0 in this, this one does not go to 0 it becomes v naught because this factor becomes 1, this factor becomes 1 and you have v naught which is correct which is what we did. And q2 dot at time t equal to 0 will be 0 because of this factor so the q2 is at rest at time t equal to 0 which is consistent with what we expected. Now let us see what is happening as time goes on, as time goes on.

So, initially both are at rest which is fine, also note one thing before I say more, this one these two I mean all the four basically, these are more, these are like oscillations with frequency omega bar. But the amplitude here is changing with time, so you can think of an envelope within which these oscillations are happening that is what you must have encounter earlier also in some other context like in your waves and oscillations or optics, that is the same thing.

So, this is an amplitude which decreases with time and within which these are the rapid oscillations are happening. See this will decrease slowly with time because the epsilon here is small so the change in the amplitude with time is small. So it also oscillates the amplitude also oscillates. So you can see that at t equal to 0 q1 dot is something and q2 dot is 0. So the first pendulum is moving with the velocity v naught and the second one is stationary.

And as time goes on and you move to time t equal to let us look at here, here, pi by 2 epsilon, I should use the same epsilon, I should use the same epsilon so pi by 2 epsilon then this will become cos of pi by 2. So which means that by the time you reach this time t equal to let us call it capital T, this envelope has been shrinking because it starts with the its maximum value at t equal to 0, cos 0 is 1 and it keeps going down with time and eventually the envelope goes down to 0.

And, your q1 dot is 0 then, but in the same time this envelope has been increasing because it starts at 0 the amplitude and as time goes on it increases and but when you reach at sin of pi by 2 which is what it is this is doing this will envelope will take its maximum value. And the q2 dot will be non-zero, you can find out what it will be but this will be non-zero. So when you arrive at this time the first pendulum has stopped and only the second pendulum is moving and as time progresses this will keep repeating.

So after another time pi by 2 epsilon has all passed again, the first one we will start oscillating and the second one will be at rest. And, this is what happens here, so what you see is that the kinetic energy is oscillating between the two pendulum. And, this is what you call the phenomenon of beats. That is nice and we will continue with some more discussion on small oscillations in the next video. See you then there.