Introduction to Classical mechanics
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Lecture 22
Coupled pendulums, normal modes

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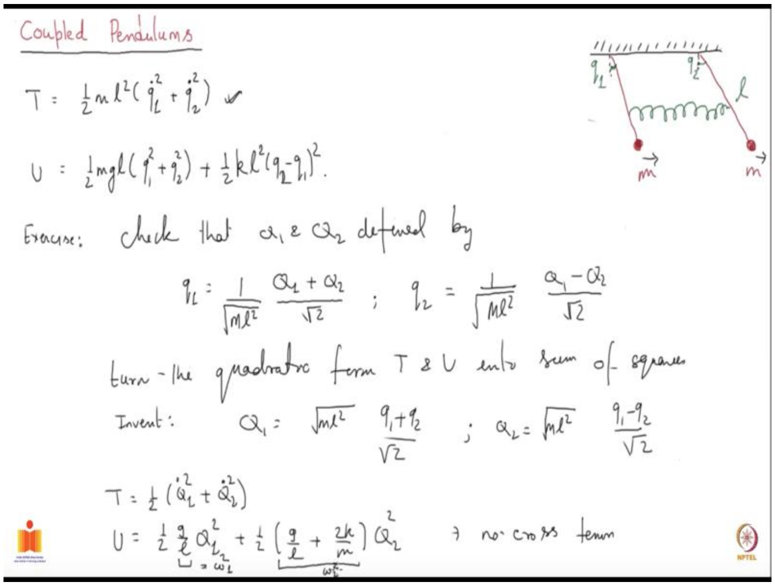

Hello, last time we were talking about a triatomic molecule and we were looking at its longitudinal vibrations meaning vibrations of the molecule along its configuration meaning the line along the line over which its atoms are placed. And we looked at the normal modes and I was also mentioning that I will take up further on molecules. But, I have changed my plans I think it will better if I take one more example of another system before we look at molecules.

Or maybe it will be better if I skip molecules all together and you can study more about it, you can find good content in (()) (1:02) and (()) (1:04). Anyhow, so for now let us take up a next example of this and we are looking at coupled pendulums now. So that is what we will do, so let draw the setup of a couple pendulum. So, we here have two pendulum which are couple by a spring, so let me draw this.

So, this is the ceiling from which I will hang two pendulum, this is the vertical direction and let me try to fill in some color to make it brighter. Let say you have one pendulum with some mass here and another one here let say the masses are same, both the masses are same. And, then there is a spring which connects them, so you have a spring here which connects the two. And, the angles here I will denote by, these angles I will denote by q1 and q2.

These are, as you know our generalize coordinates, and of course as these two pendulums move this spring will get stretched or compressed and it will store potential energy. Let say the length of the pendulum is l, both of them are of same length, masses are m as I have said, that is fine, it is good. So, if you write down the kinetic energy and potential energy you will find the following. The let me change the color back to black, this will be the kinetic energy here.

So T half m l square q1 dot square plus q2 dot square, that is good and the potential energy will have two components one coming from these masses getting moving in the gravitational field of earth. So that gives you half mgl q1 square plus q2 square, this you can easily find. And then, the other piece comes from the spring the, because the spring is getting stretched and that gives a contribution of, so you know your half k x square that is the same thing here.

So you get half k, so k is the spring constant l square q1 minus q, I can write q2 minus q1 it does not matter, square. So these are the two quadratics forms that we have and we would like to put as sum of squares which we have being doing earlier. And that is what you do by normal coordinates. But given the experience we have already with the previous example we can almost guess what the normal coordinates would be and completely avoid doing the algebra of finding out the Eigen vectors of I mean finding out all the Eigen values and then building up the transformations which put them in quadratic, in sum of squares.

So it will I will give you an exercise, check that Q1 and Q2 define by the following relations. So this is guess work and it work. So what you can guess is the following, you can guess that if I define Q1 to be Q1 plus Q2 over square root of 2. Of course this whether this is square root of 2 and some other coefficients is one can easily fix. But the point is that you can guess that I should use such a such a combination Q1 plus Q2 and Q2 to be 1 over ml square Q1 minus Q2.

So check that Q1 and Q2 define by these two relations, turn the quadratic forms T and U into sum of square, turn the quadratic forms T and U into sum of squares. All you have to do is put these two definitions in T and U and check that indeed you get sum of squares. Now of course if you invert these relations which I have written then you will get the following, you will get Q1 as ml square q1 plus q2 over square root of 2 and Q2 to be ml square q1 minus q2 over square root of 2.

And so I mean I could have started by saying that this is what you can guess that I should define a Q1 to be sum of this and Q2 to be difference of this. And you can guess this because let us say if you define it this way I mean we are doing via what I am trying to say is how to guess that these will be the modes without doing the algebra. So let say if you define these as the modes then when Q1 is, Q2 is 0 then you will have Q1 equal to Q2, meaning both the pendulum here this guy and that guy will be moving in the same directions so if this goes in this direction that also goes in that direction.

And, clearly it looks like that will be a mode, because this entire setup will be then oscillating with one frequency because this spring will not get stretched or compressed it all. And, these two guys will move as if there were no connection between them and they will be moving at the same frequency, so you will have a normal mode. So that is how you can guess this part and clearly if you have guessed this you can also think that maybe I should put Q1 plus Q2 and you will realize that indeed that is important if you wish not to disturb the T. So please check this and try to see that you can I mean it is kind of understandable that these are the choices which you should make. And if you do not like you do the algebra which we have done earlier and arrive at these.

So once, you do the exercise you will be convince that these are indeed the right coordinates and in these coordinates I will get the following. So check that you get T equal to half Q1 dot square plus Q2 dot square, so indeed it is not disturbed. It was anyway sum of squares here, so we had to ensure that we do not disturb it. But the Q now becomes a sum of squares and this is what you should verify Q1 square plus half g over l plus 2 k over m Q2 square and there is no cross term, no cross terms involving Q1 and Q2.

That is good, so what are the characteristic frequencies of the system, these are the characteristic frequencies. So square root of this let us, we will call omega 1 and square root of this thing we will call omega 2. There are our frequencies of the normal modes, this is omega 1 square and this is omega 2 square. Let us go to the next slide.

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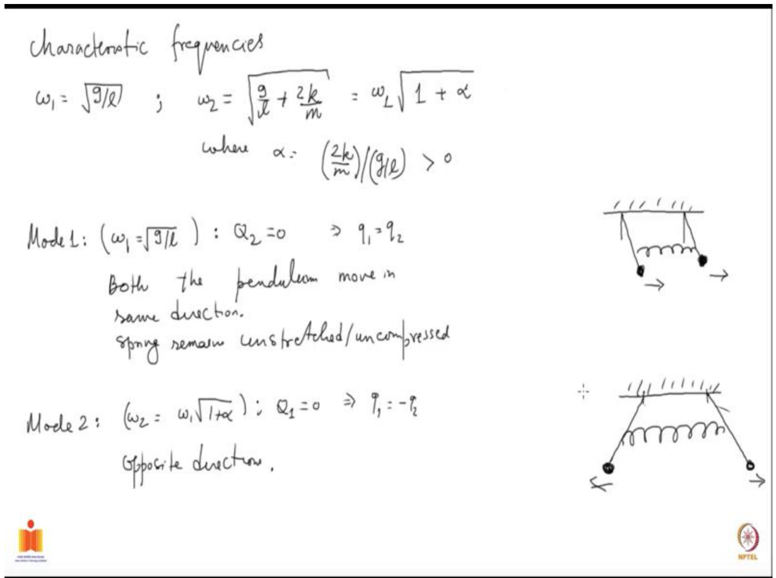

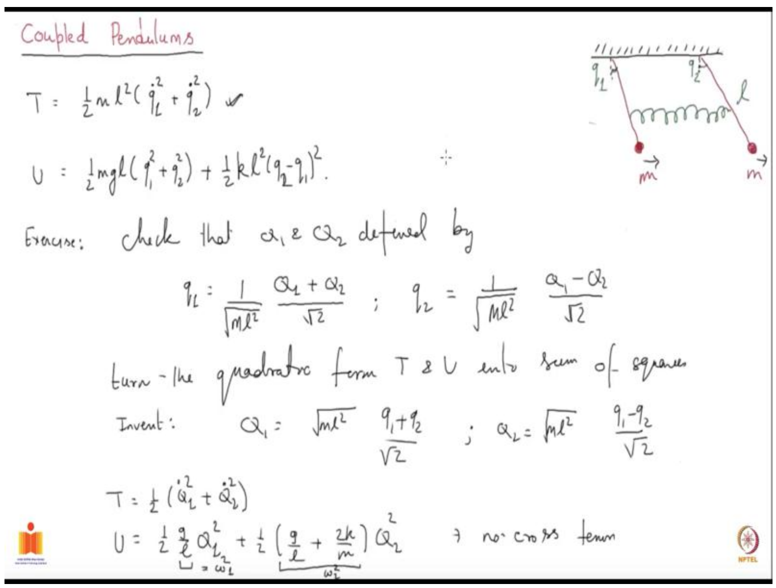

So our system has the following characteristic frequencies. So one is omega 1 g over l that you know that is of a simple pendulum. And it is not surprising that this is what you get, as I explained couple of minutes ago and omega 2 is g over l plus 2 k over m square root. Which is again you can write as this omega 1 which is what I have taken out 1 plus alpha where alpha is 2 k over m this entire thing divided by g over l and alpha is clearly greater than 0.

So one thing you notice is the frequency omega 2 is larger than omega 1 because alpha is greater than 0 and you may wonder why that is the case? And here is you see that, see when Q2 is at rest that mode is not activated only Q1 is there. So Q2 at rest means Q1 is, small q1 equals small q2 meaning these two pendulums are moving in the same direction together without stretching this, that is the mode.

And clearly, when Q1 is 0 and the other mode is activated these two are going in opposite directions because when Q1 is 0 you get Q1 small q1 equal to minus small q2. And in that case in that mode this spring is getting stretched or compressed. So more potential energy is stored in that mode, so when that mode is active there is more potential energy that you can store. One is in the gravitational potential energy of these two masses plus the, plus the potential energy stored in the spring. And that is what leads to higher frequency in this mode.

That is good now let us write it down anyway whatever I have said so mode 1 omega 1 is g over l square root that is our mode 1 which means that my Q2 is 0. Of course this is not active, so the frequency of this one is what I am writing down and this implies that q1 is equal to q2 as I said. And in this mode both the pendulum move in the same direction, spring remains un-stretched or uncompressed. So here what it looks like which I have been saying now for some time.

So here both angles will be same, does not look like and they are moving in the same direction. That is what you have mode 1 and then mode 2, your omega 2 is omega 1 1 plus alpha. As I said before Q1 will be 0 and you will have q1 equals minus q2, moving in opposite directions. And here the situation is this, it does not look like straight. So here you have the vertical so let say this guy is going this way, then then this guy will go this way in a symmetric fashion and of course the spring is stretched or compressed when they are returning backwards they will it will get compressed.

So this is your mode 2. Also note that when you have your system oscillating in one of the modes, the visual that you get in front of you, so if you are looking at this system oscillating one mode what you see in front of you is a shape which does not change at a time because all parts are moving with the same frequency. I will clarify this point a little in a bit more. But a what I am saying is, as time goes on if you have your system oscillating in one particular mode then the shape which is in front of you does not change with time it just become smaller and bigger or remains the same that is what will happen.

So for example, if you look at this one it will look identically like this at a later time also. So for example it will become this pendulum will get here that pendulum will get here at a little time but the shape looks the same. It is not that this guy will be two meters away and this guy will be one meters away, that will change the shape.

But these two, but this will look the same at all the times it will be just looking bigger or smaller where it is more out or less out that is what is going to change with time. So the visual perception of what you have in front of you in a particular mode does not change with time. It just only becomes bigger or smaller and the reason is because all the parts are moving with the same frequency when you are in a normal mode. And the reason is because your, let us say when you are in a particular mode Q1 for example in this case.

All the small q's they all oscillate with the same frequency, right that should, that you should be able to immediately convince yourself because the small q's are just a linear combination of the big Q's for example here this q1 is just a linear combination of these. q2 is linear combination of this. So if let us say q2 is absent here because you have put it to rest then whatever frequency q1 has, the omega 1 is the same thing q1 will oscillate with and the same for q2. So, in the normal mode all parts oscillate with the same frequency but some of the parts maybe at rest.

For example in the case of triatomic molecule when you had a symmetric mode the center particle the one with mass capital m located at x2 that was not moving. So either all the pats move with the same frequency or some of the parts maybe just at rest. But in general if you are not in a, if your system is not oscillating n a particular mode then this is not true. So different parts will be moving will different frequencies.

Anyway, we will come back to all this in more detail later but for now let us continue with this example. So good that we have found the characteristic frequencies by guessing the normal modes, normal coordinates. Now this now phenomena, which happens in this example. Imagine this spring which connects the two pendulum here its very weak very feeble. So that the interaction between these two pendulum is very small.

Which means let us say you start with your pendulum both of them hanging down straight nothing happening and at some point you decide to nudge one of these gently. So that you are doing small oscillations that will not immediately disturb this because the coupling is very weak and that is what we want to look at and we will see something very nice that comes out of that algebra which is usually demonstrated in classrooms, I will show this.