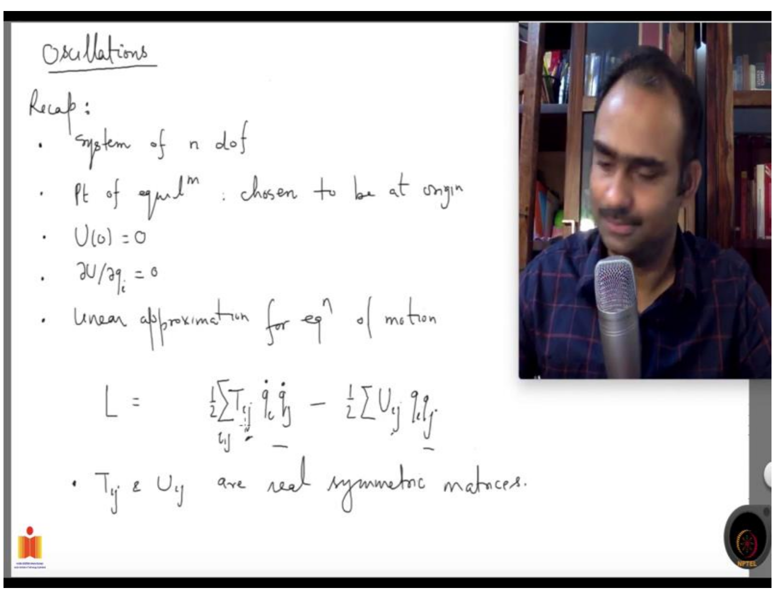
Introduction to Classical Mechanics
Professor Dr. Anurag Tripathi
Indian Institute of Technology, Hyderabad
Lecture 19
Oscillations, Normal Coordinates

Hello, last time we started looking at General System of n Degrees of Freedom, a lateral system which we had defined last time. And we assume that there is a point of equilibrium and we were expanding the terms in the Lagrangian the kinetic energy term and the potential energy term, around that point of equilibrium. Now, this point of equilibrium could be stable could be unstable and that information will be encoded in the potential energy matrix the Uij term the Uij's which we wrote down last time.

And then we did some truncation in the Lagrangian, so we approximated I mean we did a Taylor expansion around the origin where we had chosen the point of equilibrium and then we truncated the series in such a manner that the equations of motion that you get are linear and that was left as an exercise to be done by the viewer, which maybe I can do once maybe in the next lecture. And then we wrote down Lagrangian with under that approximation. So, let me write down by summarizing what we did and then we proceed from there.

(Refer Slide Time: 01:44)



So, we are looking at oscillations. So recap, first we are looking at a system of n degrees of freedom and then we assume that there is a point of equilibrium of the system and we have chosen that to be at the origin. We also put the potential energy to be 0 there, that is the choice we made, so U at 0 by this 0 I mean all the coordinates take value 0 has been chosen to be 0 and the fact that that point is a point of equilibrium meant that the derivatives del u over del qi they vanish there, that is what we derived is a condition.

And under linear approximation vision for the equations of motion I believe that you have already done that exercise, we wrote down that the Lagrangian becomes the following, it becomes half Tij qi dot qj dot and you have summation over all i and j running from 1 to n and then you have minus half Uij qi qj, where Tij's and Uij's they are constants, now we have already expanded about the origin and we also know that our T sometimes I will write T and U to mean these two these two things or maybe let me just write down Tij and Uij are symmetric are real symmetric matrices, that is good.

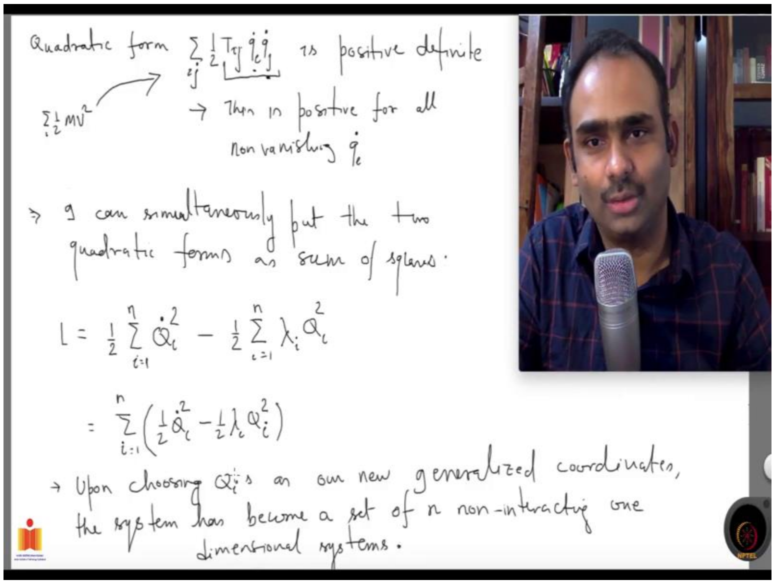
Now, also recall that I did not make any assumption about this Uij, remember this was appearing as a second order term in qi's, the first term was U of 0 and the second term involved del u over del qi this one which went to 0 and this was the first non-zero term, so depending on what these coefficients are doing, what values they take the point could be a stable equilibrium, unstable equilibrium or whatever. So, that information is coded in here.

So, as far as this where we stand I am not necessarily making an assumption that I am near a stable equilibrium, this could very well be a unstable equilibrium point, but for now we will pretend as if we are near a stable equilibrium and proceed and later we will also talk about unstable equilibrium. Let us, see, so I was saying last time before closing that the moment we see these two quadratic forms, so as you might have already noticed that we have two quality forms here and immediately we want to ask whether these two quadratic forms whether we can say whether these are positive definite or no such property can be ascribed.

So, you want to know about their different properties. Now, what I am going to show is which is quite easy, that the first form, first quadratic form this one is a positive definite quadratic form and it is quite easy to understand, what do I mean by saying it is a positive definite quadratic form, it means that no matter what these column vectors and these row vectors, see this you can write as a row, so you can write as a T as a matrix, a row from left and a column from right, that is what you can do.

So, what we are really asking is, whether this form takes a positive value no matter what row and columns you choose I mean the row and the columns are related because they are transpose of each other, if it is never 0 and if it is always positive then it will be a positive quadratic form and this what I am going to argue, which is fairly simple, maybe I can write down.

(Refer Slide Time: 07:46)



So we are looking at the Quadratic form, q Tij qi dot qj dot that is what we are talking about and you have a summation over i and j. Now, remember what how you arrived at this term, this gives you the I mean this was the kinetic term in the Lagrangian to begin with and you have done some approximation and this is what you have. Now if this represents kinetic energy, then if these velocities are not 0, if your system is really if your coordinates are really moving they have some velocity, then the kinetic energy has to be positive, no matter what those velocities are.

Because it is a kinetic energy, remember kinetic energy is this, it is a sum of all these positive numbers and that is what has and this is what has become this, so no matter what qi dots you take this is always going to give a positive value. So, this is positive for all non-vanishing qi dots, the only way this will give you a 0 is, if the velocities of all of them all of the generalized coordinates are 0 that is the only case in which you can say the kinetic energy is 0.

But then that is not how you talk about a quadratic form, you talk about non-zero vectors, so whenever you are vector qi dot is non-zero this will be positive, so I have argued that this quadratic form is positive definite. And this is very very nice, because the moment I can say this there is this positive definite I go back to what we were talking about simultaneous diagonalization of two quadratic forms and see that I will be able to not only put this in diagonal form with unit coefficients, but also simultaneously diagonalize the quadratic form corresponding to the potential term.

So, by this I mean I can simultaneously diagonalize or let us simultaneously put the two quadratic forms as sum of squares, sum of squares, so my Lagrangian now becomes the following, so my Lagrangian becomes the following. So, the fact that I can put as sum of square means that I will be able to do a transformation and go from the small q's here to some new set of coordinates, let us call them capital Qi's, such that the following holds true, so sum of squares.

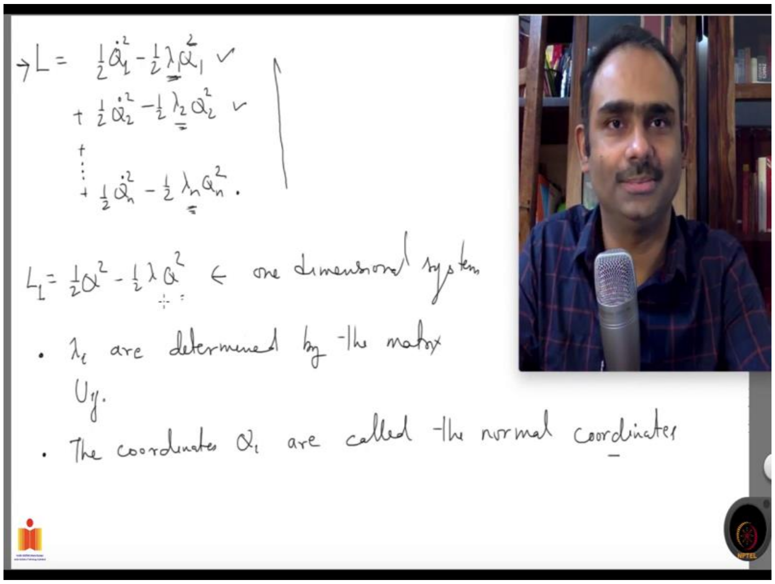
So, first one will be with unit coefficients I will put half anyway outside that I can written this no problem, so I am saying this piece, this piece is being put as some of quadratic squares with unit coefficients, i now there is no j because these are squares Qi dot square, 1 to n your n degrees of freedom minus half, so under this approximation which we have made the linear approximation I have been able to write down my Lagrangian using a new set of generalized coordinate capital Q's in this form and this is very very nice result, do you understand why this is very nice, what we have achieved, let me put it slightly differently, let me write it in this way, i 1 to n half Qi dot square minus half lambda i Qi square.

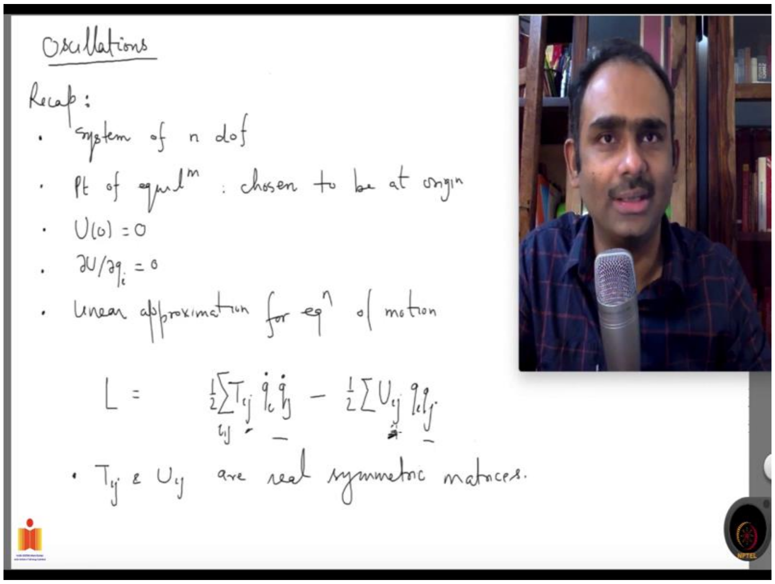
Do you appreciate why this is so nice, you see what it means is that my system upon choosing this new set of generalized coordinates appears as a collection of one-dimensional systems which are not interacting with each other, let me write it down, this means we have cast our system now this is not a nice intense, let us say upon choosing Qi, Qi's as our new generalized coordinates, the system has become a set of n one-dimensional non interacting systems, has become a set of n non interacting one-dimensional systems.

And why are they not interacting? Because there are no terms which couple qi to qj, so there is no cross term between qi and qj. So, there is nothing like q1 and q2 are couple, so when you are writing down the equations of motion for qi it will not get any contribution, which will have a q j appearing in it. So, let us say you are writing for q1 from this Euler you get the right down the Euler Lagrange equations, let u say for q1 that will not involve q2. So, meaning no matter what q1 q2 is doing, q1 does what it has to do.

So, they are really independent of each other and that is what I mean by saying that these are n non-interacting one-dimensional systems. So, actually this is very very nice because one-dimensional systems are easy to understand and this is what I have achieved, let me see what else I wanted to say. Now, our task is to first identify what kind of system this one-dimensional system these are, they are all going to behave same way. Then the only thing which will be different between these between any two different system, let me write down again, let me write it down, it will be 0 to say what I am saying.

(Refer Slide Time: 17:48)





So, L is let us say half Q1 dot square minus half lambda 1 Q square Q1. So, that is one piece, then you have half Q2 dot square half lambda 2 Q2 square and so forth, that is what our Lagrangian is. So, if I know about this system this one dimensional system, let me call this L1 then I will know everything about this system also, because it is just n copies of this system. The only thing different between this and this is the lambda 1 here and lambda 2 here and all these coefficients.

Other than that, there is no difference between these different dimensions one-dimensional systems. Now, what controls these lambdas? It is controlled by the matrix U, where is that, here, see those lambdas are the eigenvalues of this matrix U, so it is the matrix U which controls them, so we would be interested in knowing more about Lambda, so lambda i are controlled or determined let us say by the matrix Uij that is one thing, so we need to figure out what are the possible lambda i's and of course this will depend on properties of Uij which is also tied to whether the equilibrium is stable or unstable.

So, that is one thing, we want to see in the next one next video and let me just tell what these coordinates are called. These coordinates Qi are called the normal coordinates that is good. Now, let us say you are given two different system and you choose some coordinates some normal coordinates, which I have just now mentioned here and put the Lagrangians of both these two systems in this form in this manner, now if you do so, what will be different between these two systems? It will be just the lambda i's.

So, it is just the matrix U which is going to differentiate between this system between one and between and the other one. So, we need to know a little more about these lambdas that is what we will do next time and I hope you already know, but anyway I will do it next time that this is the Lagrangian of a simple harmonic oscillator if lambda is positive. I believe I have derived this or talked about this earlier, but in case not I will I will do it in the next video.

So, as far as if lambdas are positive then this system has become a set of n non-interacting one-dimensional simple harmonic oscillator, which is a very nice result. So, we will continue next time with more details, but you can already start imagining the following, let us say I can choose to set my system in motion in such a way that not all the normal coordinates are moving, so not all of them start moving only if some of them or let us say only one of them starts moving and all others coordinates remain stay put they do not move.

And this way you can isolate the motion of one of these oscillators. Meaning you can excite that we use the word excite, you can choose to excite one of the oscillators and keeping all other oscillators silent, they do not move, they do not oscillate, so this you can do. We will talk about these things in more detail in the next video. See you then.