Introduction to Classical Mechanics
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Lecture 18
Small Oscilations

Hallo, in the last two videos we were looking at some of the mathematical aspects that which we are that we are going utilise in this course. Mainly they covered matrices and quadratic forms. We saw that if you have two quadratic forms, and one of which is a positive definite, then we can put these two quadratic forms as sum of squares and one of them which corresponds to the positive definite quadratic form that can not only we put as a sum of squares but also the coefficients which appear in front of these squares, they can be chosen to be unity.

So, that is what we were talking last time and today we are going to start a new topic which will spend some time with and this is oscillations. So, that is what we are going to do now.

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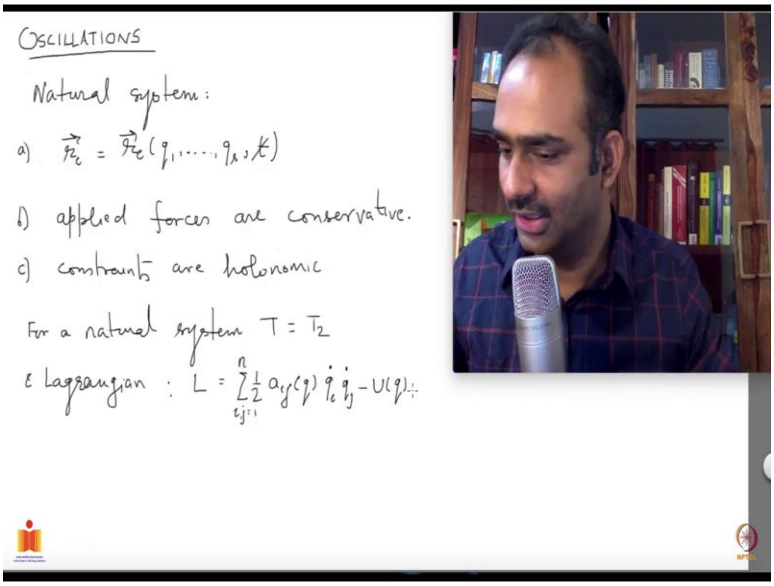

So, I will start this lecture by defining our natural system. So, generally when we are dealing it problems, it will often happen that we do not need to have time appearing explicitly in the transformations when you go from Cartesian to generalised coordinates. So, the time does not appear explicitly there that happens very often. Also the applied forces, the forces that are under your control not the ones arising due to constants.

The applied forces will be conservative. This also will be a frequent occurrence and that constraints will also be very often holonomic, meaning that you will be able to from the outset get rid of the degrees of freedom which are not independent. So, you will be able to utilise the equations of constraints and then write down your system only involving independent coordinates.

So, these 3 conditions are very frequent and if a system obeys this or if a system is obeying this then you say that the system is a natural system. So, let me write down what I have said just now. So, natural system and the conditions are the following, a: The (trans) or maybe I should write even without this so, what we are saying is if you look at q relation between the Cartesian coordinates and the generalised coordinates. This does not involve time explicitly, that is one.

Second is forces are conservative. And third, constants are holonomic. If these 3 conditions are falling true, then we say that the system is a natural system. And for a natural system, clearly the kinetic energy which is in general a sum of 3 terms, one is of degree 0 in generalised velocities, t0 that is what we have been saying. Another is t1 linear in generalised velocities and t2 quadratic in generalised velocities but for a natural system, you will have only t2.

For a natural system, the kinetic energy is only condense only t2. And the Lagrangian will be of the form of the following form. So, your L will be so let me write down explicitly t2 it will half aij which in general depends on all the coordinates qi's which I am denoting them by q and then you have qi dot qj dot minus the potential energy which depends on again all the generalised coordinates. Let me put a summation 1 to the total number of degrees of freedom.

That is the definition of a natural system. Now, we want to talk about oscillations and the generic situation is the following. Usually, your system would have a equilibrium configuration. Meaning, if you put your system at rest, so let us say all the generalised velocities are 0 and if you put your system in such a configuration and if system remains in this configuration meaning it always I mean, it always generalised velocities remain 0 with time.

So, they do not change, they keep remain 0. Then you say that the system is in is in an equilibrium configuration, configuration maybe stable or unstable but what you except that the system is not going to move away from it if that is an equilibrium configuration. So, let me write this down.

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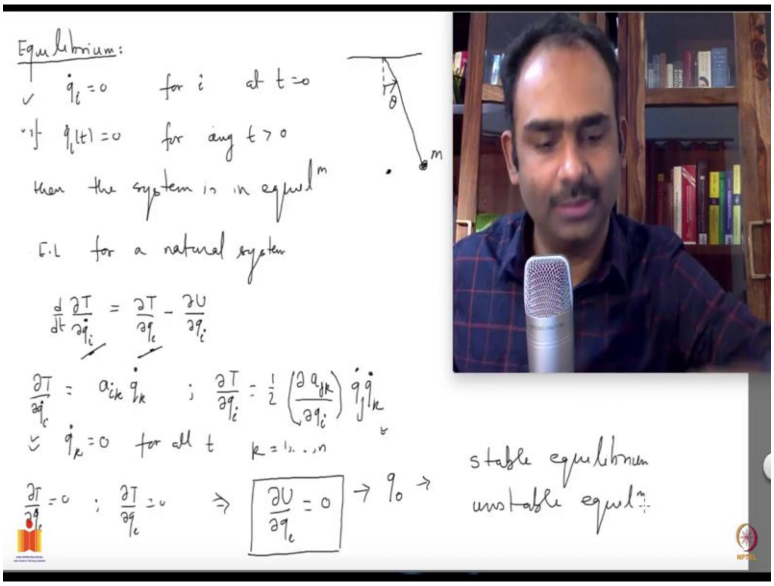

Equilibrium, so what I am saying is let us start with all the generalised velocities qi all of them are 0 for all i qi dot generalised velocities they are all 0 and if so that is how you start and if qi dot remain 0 at any other time, so let us say this you have started at t equal to 0. So, t equal to 0 you have put the system such a form that all the generalised velocities are 0. And if at any later time this still remains true, then it means that your system is residing at an equilibrium point.

For any t greater than 0, then we say than the system is in equilibrium. Let us take a very similar example. Let us say I have a pendulum and your generalised coordinate is theta in this case. Now, if you take this mass and put it here vertically downwards and put it at rest, then it will remain there, it will not move. So, at any later time t it will still be lying there. But if you give it some velocity there at the at the lowest position, then of course it is not going to stay there. So, clearly you see that this point will be an equilibrium point because this connection is falling true.

So, that is good. Now, we need to find out the condition for equilibrium. So, this is what we expect for I mean, this is the definition for the system being in equilibrium but what else needs to be satisfied for this to hold true? That is what we want to look at. So, we start by looking at Euler Lagrange equation because that is what tells you how the system is going to evolve with time.

So, Euler Lagrange equation for a natural system is d over dT, here you have generally del L over del q dot but L has a U and that U does not depend on generalised velocities. This the part of definition of a natural system. So, I have only the kinetic energy term here qi dot. And then I just flip the L into T and U and written down like this. That is your Euler Lagrange equation. If you combine this two this becomes del L over del q I. Now, let us look at this thing let see what del T or del q i dot is so this piece this will be just aik qk dot.

What I have done is taken this one this this piece and have differentiated it with q one of the with the generalised velocities qi dot. So, that is what you get. Also let us see what del T over del qi is? Now, I am looking at this one. First I had looked at this one, this piece, now I am looking at this one. Now, this is del T over del qi, so the q dependence is only in a. These are the velocities. So, if I take a derivative, I get you have a half del ai let me put j. In fact I should put j and k.

I have changed the indices because they are dummy and del qi is fine. This piece and its contracted with qj dot qk dot. Now clearly if I put qk dot for all times to be 0, then both these terms are going to vanish. So, if qk dot is equal to 0, all t and k is not just one particular but all the coordinates. So, k runs from 1 to n. Then your this piece and this piece they are both 0. So, del T over del qi dot is 0 for all times and del T over del qi is also 0.

Which means that this piece is gone and that piece is gone. Now this entire equation will be satisfied. It has to be it will get satisfied only if del U over del q that also is 0 otherwise, this will not be satisfied. Which implies that the system to be in a in a equilibrium configuration, we had the condition that del U over del qi is 0. This is the condition we get. Which means we need to look at the function U and search at which point q q is q1, q2 and all, at what point that derivative is 0. And that point will be called an equilibrium point.

And clearly this is a point of extremum where the potential will take its either maximum or minimum value. So, as I said, not sure whether I said but there are 2 things which can happen. Whatever point q0 you get from here, where this condition is satisfied, this q0 could be either a stable equilibrium or an unstable equilibrium. And what do I mean by that, I mean by that the following.

So, it may happen that you put your system at the equilibrium place where this condition del U over del q is 0 and if you put it there at rest, it remains there forever. That is fine, but let us say you displace your system from that position by an infinitesimal amount. A little bit. And if you do so, the 2 things, there are 2 possibilities which can happen. One is that the system moves away from it a bit but it tries to come back.

So, it remains in the vicinity of that point q naught. It does not go away from there. So, it remains in the neighbourhood of that point q naught. If that is the case, your equilibrium configuration is a stable configuration. It is stable it remains there. Your system remains there but it may also happen that even though you can place a system there at rest, but if you give a small infinitesimal nudge to the system which takes it away from that point, system does not come back to this place. So, it goes further away, it does not want to return there.

For example, you can a very simple example could be think of a sphere, if you put the ball on the top, take a ball and put on the top, if you leave it leave it at the rest state, it will remain there but if you give it a slight displacement, it is not going to come back. But if you take a bowl and then you put the thing at the equilibrium place at the centre, if you leave it, it remains there but if you take it away from there, it remains in the vicinity of that point. It does not go away far. So, that is what generally stable and unstable equilibrium are. May be I should write that down.

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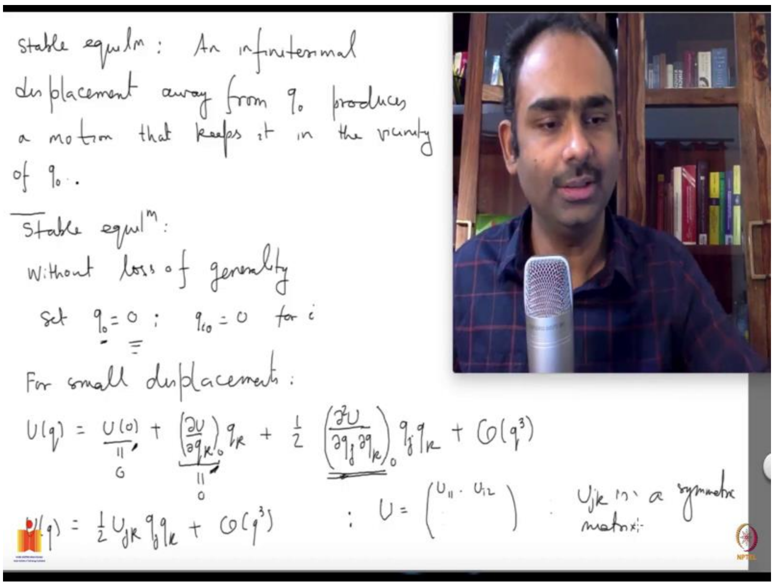

What happened? Stable equilibrium. And infinitesimal displacement away from q0 produces a motion that keeps it in the vicinity, in the vicinity of q naught. And clearly if you have unstable it will take it away from q, it does not return back. It just goes further away and depending on what is happening further away from that point, q naught your system may keep going away forever or let us say it has another maximum minimum so it can do something else.

But as far as the neighbourhood of point q naught is concerned, these are the statements that we can make. Now, of course we are going to be interesting in stable equilibrium. So, henceforth I will talk only about stable equilibrium stable equilibrium. Now what I can do is without loss of any generality, I can choose that point where you have the equilibrium, the stable equilibrium as to be the origin of your coordinate system.

So, I say that all the q's are 0 there, basically, choosing the origin of your coordinate system. So, this is what I am going to do. So, without loss of generality I can set q naught to be 0 with basically means that I am saying qi. Let us say, how do I put it? qi. qi 0 equal to 0, for all i. Here q naught stands for the entire set of coordinates, that is good.

Now I am going to be interested in small displacement around q naught. See, if your displacement is large, then just knowing about the potential in the vicinity of q naught is sufficient. Then what is happening globally will determine the motion. For example, if there is another minimum somewhere and if you displace it by large amount, your system may end up in this other minimum. So, we are interested in small displacements.

And if I am looking at small displacements, what I can do is, I can take the potential and do a Taylor expansion around this point q naught which I call the origin now because I have said that already to be 0. So, I will do a Taylor expansion for small displacements I will do Taylor expansion so, U of q is U of 0 I am expanding around the point 0 plus del U over del qk evaluated at the origin qk. That is the displacement from the origin qk is now displacement from the origin.

So, that is one advantage of putting everything at origin. So, I do not have to write qk minus something. Plus half you should take a second derivative of U with respect to qj qk and again this thing should be evaluated at the origin. So, these 2 numbers are now or these 2 quantities are number, they are not functions of qs anymore. They are just some numbers which are determined by your system.

And then you have qj qk plus you will have all other terms. So, I will write them as order q cube let me just put q. So, I am by this symbol I mean I am dropping I mean, all these terms which I am not showing, they are all order cube or higher, depending on the system some of the terms may vanish also. Some of the orders may be absent. But generically this is what the statement would be.

Now, I can do another thing I can set U of 0 to be 0. See, potential you can the absolute value of potentials are not important that you I believe already you know. It is only the relative difference which are important. So, I can set that the potential energy of the system is 0 when it is at the equilibrium configuration. So, that choice I can make, so let me make that choice and I set this to be 0.

Also because I am saying that my system is in equilibrium, I have already seen that that translates to saying that the first order derivative U vanishes. That is what we have seen a couple of minutes ago. So, I put this to be 0. See, this follows from our requirement of equilibrium and this is a choice I have made. You are free not to choose this, you may keep it something else non-zero nothing will change.

But anyway things look simpler if I made this choice and with this choice I am left with this term which is quadratic in q and higher order terms. So, with that I get U of q I will write this as I belief I think it will be half, this I call Ujk. So, these are just numbers qk, qj, qk plus higher order terms which I am denoting by order q cube. Now, clearly the matrix elements Ujk so, you can think of this n cross matrix see your if you look at j and k they run from one to n that is the number of degrees of freedom you have.

So, this Ujk if you look at them this is this are n square numbers total n square number and they form a matrix and the matrix we call U whose entries are this U 1 1 and so forth U 1 2 etc and clearly the matrix is symmetric because of this definition of it. If you interchange two derivatives it does not change because this derivatives differentiations commute. So, U is a symmetric matrix is a symmetric matrix real symmetric matrix. Now, let us look at t kinetic energy term.

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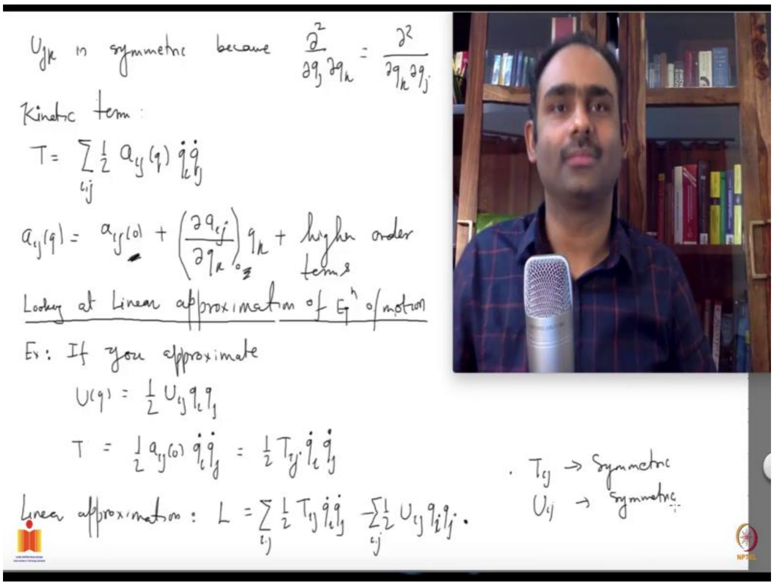

Before I do that that is fine. Now, look at, maybe I should write it down Ujk is symmetric because del qj del qk this order does not matter. Now, let us look at the Kinetic term and what did we have? We had t equals half and you had a summation over i and j aij it depends on q in general and qi dot qj dot that is the form for a natural system as we said. Now, what I am going to do is I going again do a Taylor expansion around origin where your system is in equilibrium.

So, I write it as aij q equal aij of 0 plus over delta qk evaluated at the origin qk plus higher order terms. So, all the terms will be there that is good. Now, that is fine, now let us look at our Euler Lagrange equation of motion. So Lagrange is there but I want also look at the equation of motion and my intension is to see how much of this expansion.

So, up to what order I should keep both in the Kinetic energy and the potential energy terms so that my equations of motion are linear. So, I am going to be in I am interested in looking at the system in a linear approximation and the reason being that I can solve it without running into any complications though that system I will be able to solve.

So, I want to see up to what order I should keep terms in T and up to what orders I should keep terms in U so, that my equation of motion are linear that is what I want to do and clearly if I do so, it will not describe the true motion of my system but it will capture the most general features hopefully if I remain in the vesinity of the equilibrium position. So, that is what we want to do and that is what we are going to do now. So, let us look at equations of motion.

So, we are interested in looking at linear approximation of equations of motion. May be I should just tell you I will just, I will leave it as an exercise I do not wish to do that simple steps here. So, exercise show that if you if you retain or if you approximately say if you approximate the potential energy to the following where was it here if you keep only this term and drop all other terms which are here so, half Uij qi qj and drop all other terms in potential energy and if you take T to be just this first piece.

So, I put half aij and qi dot qj dot if you do so and through all this terms then your equations of motion that you are going to get for this system will be linear. And that is not difficult to check so I will encourage you to do so. So under linear approximation, I would before writing I would write instead of aij I will write half Tij. This matrix made by this elements Aij 0 these numbers I will just denote by Tij so that it sounds bit like kinetic energy matrix or sounds like antigen matrix.

So, the linear approximation any system so see I am not making any assumptions about the system, system you can approximate its Lagrange by this which is nice there summation over i and j is implied qj dot T minus U so you have half Uij qi qj and I have dropped all other terms. Let me put summation over i and j its nice. So, this will our starting point for analysing any system near its equilibrium configuration and let me just mention that your Tij this is symmetric matrix, your U symmetric and yes.

So, next time we will start looking at this system but now you can already try to bring in what we talk what we were talking about in the last videos where we were talking about some mathematics you see what we have in front of you is a quadratic form here and another quadratic form here.

So, we will ask in the next video whether this quadratic forms are positive definite, positive semidefinite this things we are going to ask and if we were lucky and if you are lucky and we figure out that may be one of them is positive definite, then we will do what we learnt last time we will put them in diagonal form and then we will somebody nice results come out that will be the goal for the next video and we will stop this lecture here. So, see you then in the next lecture.