Introduction to Classical Mechanics
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Lecture 17
Principal axis transformation

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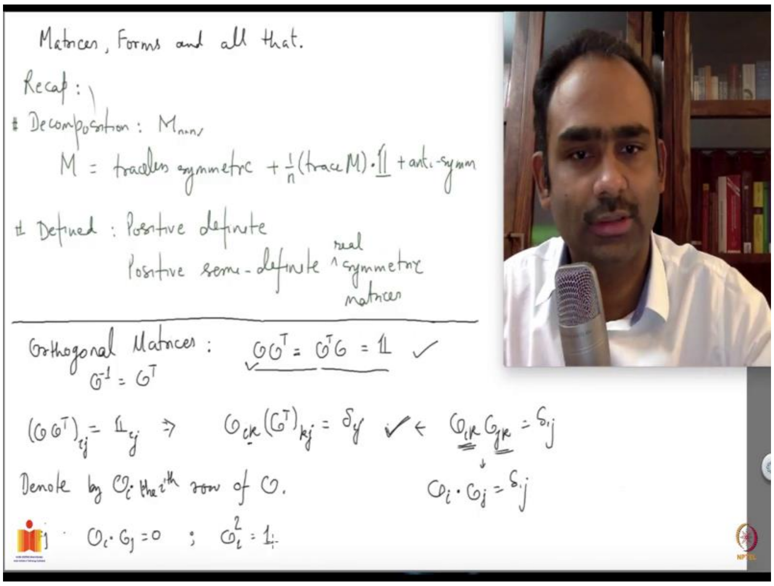

So, we denote by Oi the ith row of matrix O and I will turn this equation into into a relation involving dot products of any two rows. So, let us write this one as, I believe I wrote down this earlier also. Oik and O transpose kj will be Ojk. This is delta ij. So, if I am denoting by Oi, the ith row of i and this is a dot product between the ith row and the jth row. So, I can write this down as Oi dot Oj delta ij which is just saying that if i is not equal to j, Oi dot Oj is equal to 0 and also if you put i equal to j, then you have Oi square equal to 1.

Meaning that the, I mean any two rows are orthogonal to each other and each row represents a unit vector. So, these are all normalised to 1. So, there is a the property of orthogonal matrices. So, let us say if you if someone ask you to construct some orthogonal matrices, all you have to do is search or construct n orthogonal vectors make sure, sure that they are normalised and just arrange them in a matrix form and that is what will give you an orthogonal matrix.

And the same hold true for column vectors as well. What I have been, what I have, whatever I have said here also holds true for column vectors. I am pretty sure this was similar to you already. Now, let us move on to Quadratic forms. So, let us go to the next page.

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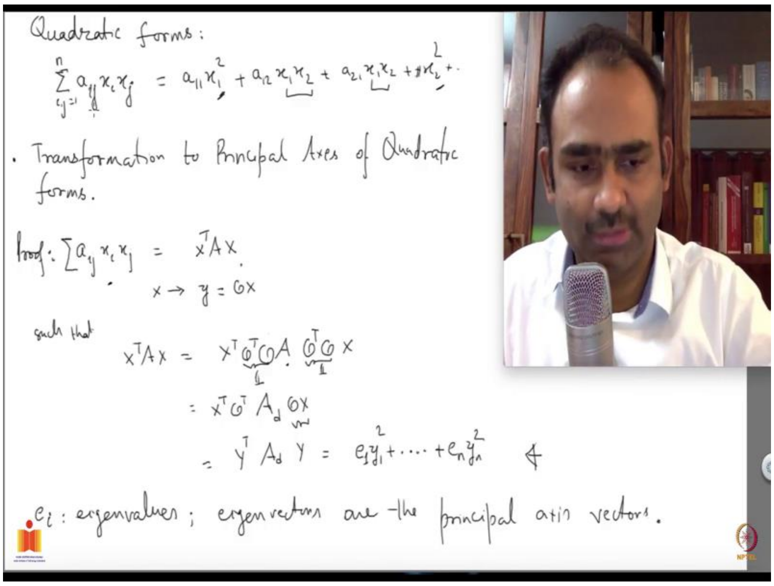

So, you are often going to encounter a quantities of this form. So, you will have let us say, what should I write here? Let us say a matrix a whose entries are ai or I think I was more commonly using alpha and beta. Let us say whose entries are, does it look good, so I will write aij, aij and this matrix is sandwiched from both sides by some row and column vectors which we denote by x and whose elements are xi and this quantity is called a quadratic forms.

So, here there is a summation over all i and j can looking at n dimensional vectors and n, n cross n matrices. So, this is called a quadratic form. And of course your aij's, they will form a symmetric matrix. Now, there is a theorem here which says I can always, let us me write it down first, transformation to principal axis quadratic forms. So, that is what we want to discuss now. Transformation to Principal Axis of Quadratic forms.

So, what I mean by this is that, we can always choose axis such that the quadratic form here this becomes a sum of squares. So, here what you have is a quantity which involves cross terms also. So, let me write it down more explicitly, if you look at it, let us say, I look at a 1 1 x1 square, that is fine. Just look a term a 1 2 x1 x2, then you have a term a 2 1 x2 x1 which is again x1 x2 and so forth.

Then you will have a term which will have x2 square something here. So, in addition to these terms which are squares, you also have cross terms. And the claim is that that I can do a transformation such that this quadratic form will appear only as a sum of squares. So, it will involve only squares like this and not involve any cross terms. And the and clearly this is going to be through orthogonal transformation because your a has to be a symmetric matrix.

And you already know that you can diagonalize n symmetric matrix by real orthogonal transformation. So, this not to surprise that we can put this quadratic form into sum of square. Anyway let us do it. So proof so the proof of the theorem is the following. (()) (6:34) but anyway it will help to, so let us say aij xi xj, I can write as let us say I denote to vector x which is a row vector, matrix A is the one which corresponds to these entries small aij. Let me put transpose here x.

So, that is what we have and I am saying I can do a transformation from x to y which is related to your x column vector by some orthogonal matrix O that you can do that you can choose and I am choosing this O to be the one which will diagonalize the matrix a. So, the choice is such that this guy or this guy becomes the following. So, I can write x not small x but X transpose A x is equal to X transpose. Here is our A, here is our X and I want to put O here, O transpose here, O O transpose where O is an orthogonal matrix.

So, O transpose is identity. This is one so I have done really nothing. I have just inserted identities so, I have not changed anything. Now, this O is one, I mean I have chosen that O which diagonalizes this matrix A. So, what I am left with is X, let me skip a, let me do it in two steps O transpose and then you have A diagonal. So the diagonal matrix corresponding to A and then you have O X.

Let me, I have already defined this to be y. I should have written capital Y here. So, anyway this is Y A diagonal Y transpose. This is if you write in terms of components so y is a column vector. It will have entries y1, y2 and so forth. So, most generally. I mean in general you will be able to write this is this as the following. Let us say the diagonal entries of Ad are e1, e2 and so forth.

So, you have e1 y1 square I should remove the sigma so e1 y1 square I have assumed that none of the e's entries on the diagonal are 0 but it is not a not a problem if they are. It is not the problem if they are repeated. Even if they are repeated, it is fine. So, I can put any quadratic form into sum of squares that is what I have done which is nice which will be very useful. It is also clear and if it is not, then you sit down and convenience yourself that these ei's are the Eigen vector, Eigen values.

Eigen values of the matrix A this. And also convenience yourself that that the transformation which we have done, this orthogonal transformation that you have done puts your coordinate axis along the Eigen vectors, Eigen vectors of A. And these Eigen vectors are called the principal axis vectors, are the principal axis vectors. All good. And because A is a real symmetric matrix, all the Eigen values are real and this all your e as will be real. So, you will still be in the real space that is one thing.

Let us see what else I want to say. I hope I did mention it but if not let me repeat all the ai's, aij these are just some numbers. These are all constants. So, now I want to say a little more about this. You can imagine that if I do a little bit more, if I do further transformation, I can not only this in a form containing only sum of squares I can just do away with all the eis so that the all the coefficients are unity. That is what I want to do now.

And of course for that I have to do a transformation which will not be orthogonal. You will see that in a moment.

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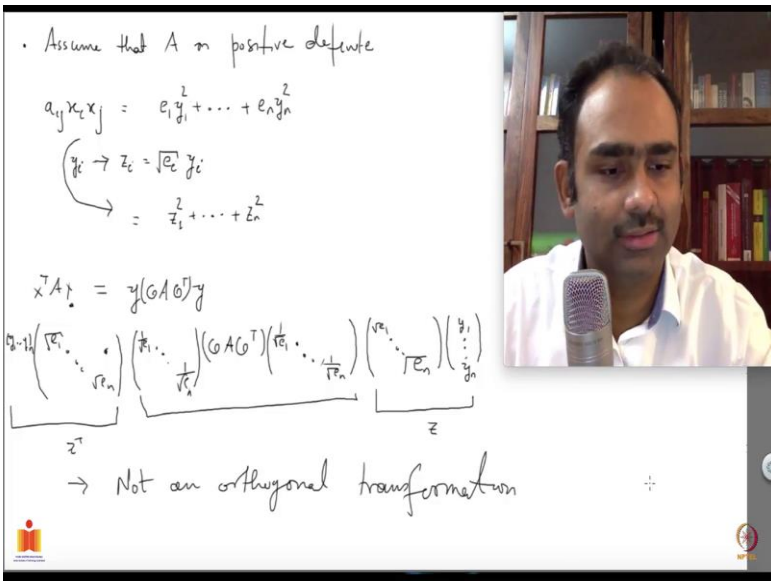

So, I want to make an assumption that matrix A is positive definite. If A is positive definite, then you will have a n Eigen values which are non zero by definition. And that is what I want to have. So, I want to have a sum of squares in exactly n variables. So, I do not want to have a situation in which I have y1 square plus y2 square, even y1 square plus e2 y2 square so and so forth up to em ym square, where m is less than n. That is not what I want, I want m to be equal to n.

So, that is why I am looking at a positive definite matrix which you will and realise why I am doing this in in a short while. So, if I do so, and as I have already said that I can always put aij xi xj into the following from, summation is implied as a summation convention is being used. So, I can put these into the following. e1 y1 square that is what we have just now shown. This is the principal axis theorem. en yn square assuming positive definite.

Now clearly if I define or do a further transformation, and I take this yi's to zi where zi is related to yi by the following. If I do this then what will happen? I will basically be absorbing these ei into the yi which is what is done now. So, this will turn into, this will become z1 square and there is no problem be all of the e's are positive. So, you do not end with something complex, which is good.

Now, let us ask whether these is achieved by an orthogonal transformation. So, we have done 2 transformation, one transformation with that orthogonal matrix O which put into this form and then on the top of it we have this transformation. And I hope it is clear that this is not an orthogonal transformation. So, let us see more explicitly. So what we have done is, gone from x, let me again put more in terms of column vectors and matrices directly.

From x transpose A x we went to as I wrote before, we transform this to A diagonal which was O transpose O and then you have here y. You had Ox which is y so, you reached here. I should be writing this as, if I am not putting an index, it is clear that it is a row or a column. And then we did something which is this which is basically this transpose, this matrix you have anyway and you had your y which I will write down, let us interest the following.

1 over square root of e1, so these are all on the diagonals. These are diagonal matrices, square root of e1, en, you have y1, yn and then here again you have 1 over square root of e1, en. This is nothing and then you have y transpose, so basically the row, y1 to yn. So, from here to here, this is what is your z, this is your z that is what we wrote earlier. And this is your z transpose and your transformation is really this one.

And you see this this is not an orthogonal matrix, this one. So, this entire transformation O transpose and this thing is not an orthogonal transformation. Why is this not orthogonal? If you take a transpose of this, it will give you the same thing back because these are the diagonal entries and if you multiply together you get 1 over e1 and 1 over e2 on the diagonal entries not unit unity on the diagonal entries. So, this is not an orthogonal transformation.

And why all these is useful and interesting to me, I will tell you now. So, here is the point. In general, if you have two symmetric matrices, you cannot diagonalizes them simultaneously. You can if they commute, but if you have two positive definite forms like what you have seen just now, you have two positive definite forms, then you can diagonalizes them simultaneously. In fact what you can do is, let me be more precious.

If you have two positive, two definite, two quadratic forms, one of them is positive definite, if that is given to you, then you will be able to put both of them as sum of squares. In fact, what you will be able to do is, you will be able to put the one the quadratic form that is positive definite as a sum of squares with unit entries what I have just shown to you here. And the other one will be diagonal with non-zero, I mean with coefficients which be not equal to 1.

This is what I want to show now and that should be quite obvious to, more it should be obvious now that we can really do such a thing. So, let me go back.

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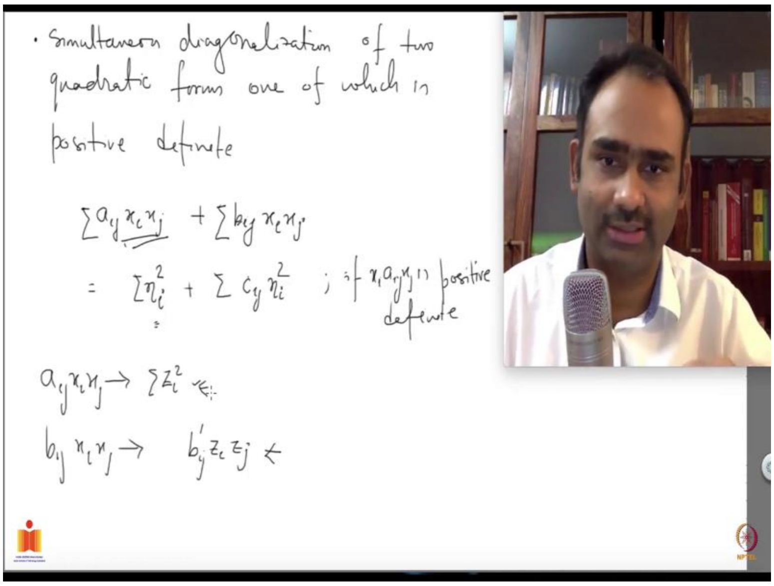

So, what I want to show is simultaneous diagonalization of two quadratic forms, one of which is positive definite. So, let us say you have the following. Let us say you are given two positive, two quadratic forms aij xi xj there is a summation over i and j and it runs from 1 to n plus another quadratic form bij xi xj again summation, there is a summation and I am claiming that I can put it as zi square summation over i. Let me put summation plus let me not use it. I will use ni or eta i.

Eta i square plus some b prime I what to use I think. Let us say cij eta i square. That is the claim provided aij is positive definite. Positive definite matrix. I mean you can also say this is a positive definite for either way. So, let me just say in words why this is going to work out. So, it is going to work out for the following reason which we will I will write down some equations to show you.

So, look at the first one. This. I have shown that in the previous last few minutes that I can out this in this form. Let us say I can put this as sum of zi squares. Let us not use eta i. So, I am saying that I can do set of transformations not orthogonal of course and it will put this into the following form, into zi square. I should now that I am putting sigma, I should put here also. This is was I have already shown in last few minutes.

Now, once you have done this, this guy bij xi xj would also transform to zi zj some b prime. This is what will happen. So, you have done a change of basis, your x's have changed to z's and the matrix bij would transform into some other matrix and it will we are calling that b prime and this b prime would also be symmetric so, you again have some situation as before meaning now you can do one more transformation and you go from z to eta such by an orthogonal transformation such that the b prime the matrix b prime gets diagonalized.

So, from z you go to eta through an orthogonal transformation which we diagnolise b prime, which will which will mean that this piece will be put into sum of squares then you may worry what happened to this what happens to this part then. Nothing will happen to this because this is sum of squares and the orthogonal transformation it will still remain a sum of squares. So, that will proof our my assertion that we can simultaneously diagnolise to quadratic forms if one of them is positive definite. Let me write some steps so, that it is more apparent but I think I have already explained this never the less let me write.

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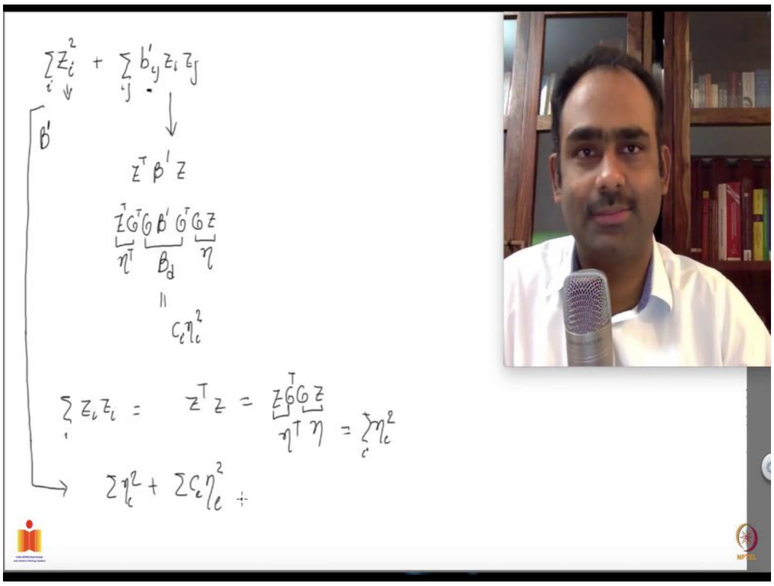

So, as I said after reducing the first piece to sum of squares with unit coefficients I would end up with bij xi, zi z zi zj and this is summation over i j and it is prime. That you agree, if you are not clear why b will be again symmetry, sit down and just check whether you are fine with that assertion. Now let us do so, the matrix I will call denote by b prime and entries by small bij.

Good, so what we have to do is now we will go from here to the following so, this is b prime z transpose z that is what you have this piece. Now, let us say b prime get disgnalised by O so, I am going to Oz O transpose and this will be an orthogonal transformation because it is real symmetric matrix b prime O O transpose z transpose. This I called eta this will be eta prime, eta transpose and this will be your b diagonal. So, good this piece becomes eta i eta j prime b prime diagonal and which I want to call as symmetric C, whose entries will be cij.

So, it I am (())(29:08) because this is already diagonal and let us call this entries as ci. So, you get eta i square ci because this diagonal there is what you are going to get and here what has happened let us look at this piece now. Your zi square is zi zi summation over i or which is same as transpose z and what is happened z we have put O, O transpose z z and this is what you have called eta this is eta transpose and you clearly see you do not pick up any questions it is just remains eta i square it still remains some of squares.

So, what you have here is the entire sum becomes eta i square plus ci eta i square. Let us so, this is what all I wanted to say in this quick review of some of the mathematical things which we will require. I think that is sufficient and we will continue with not continue, we will start looking a small oscillation the next lecture. See you.