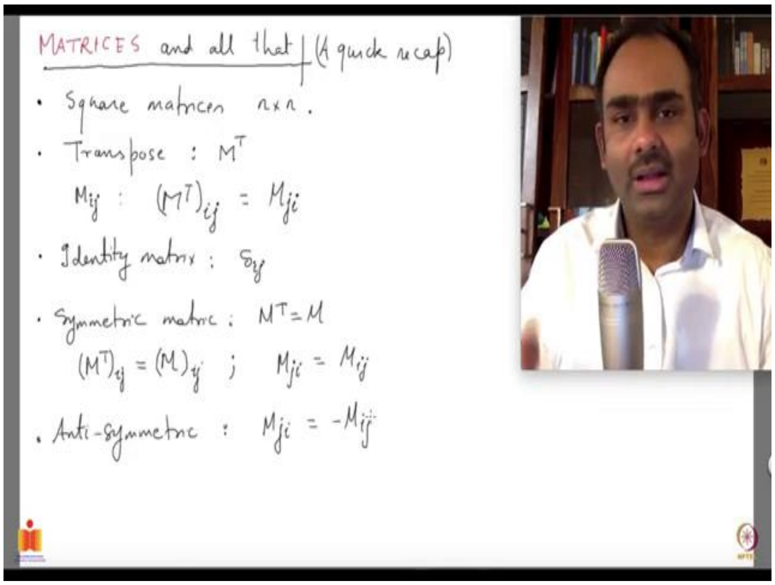
Introduction to Classical Mechanics
Assistant Professor Doctor Anurag Tripathi
Indian Institute of Technology, Hyderabad
Lecture 15
Matrices and all that

Before we go further in the subject of Classical Mechanics I would like to review certain mathematical quantities which we are going to encounter frequently in this course and it will help us keep our later discussions uncluttered, so we record everything here of all those things which will require and later we will just make a reference to what we have discussed today. So, that is the plan and here is our white board.

(Refer Slide Time: 00:44)



So, as you can see I want to talk about matrices today. Most of the things you will be already familiar with and maybe something not if but generally I will assume that you already know and I will go fast with that. But if you have not encountered certain things you will still be able to understand from his discussion. So, what I want to do is a quick recap.

So, mostly I will be dealing with square matrices so if do not say specifically that I am talking about a square matrix using that I am talking about square matrix. So, I will be only talking about square matrices in this. And in general, they will have a dimension n cross n. So, n number of rows and n number of columns that is what a square matrix is.

And as you know that to matrix you can define a transpose, so if you have a matrix m square matrix you can define a transpose, m transpose which you obtained by interchanging the rows and columns. Let us see how you write this down in index notation, so if I want to tell indices of I mean tell the element of a matrix so I specific i th row element at the i th row and the j th column, so that you say Mij , so this is the entry Mij.

So, how this Mij is related to the elements of m transpose? It is like this, so let us say you ask m transpose you take that matrix and ask what is the element in i th row and j th column and as I said it is just the interchange of row and columns of m, so it should be Mji, so this is how you should write this expression. You always have identity matrix at your disposal okay.

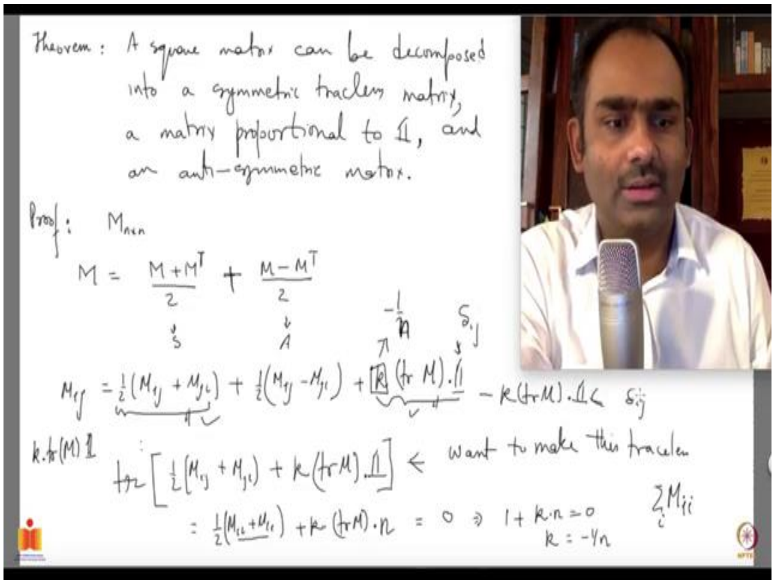
So, identity whatever problem you are doing if it involves matrices identity is there and identity is all the elements on the diagonal are 1 if you are off the diagonal it is 0 which means the components if I want to write down in components then you can represent your identity matrix by chronicle delta ij, okay that is good.

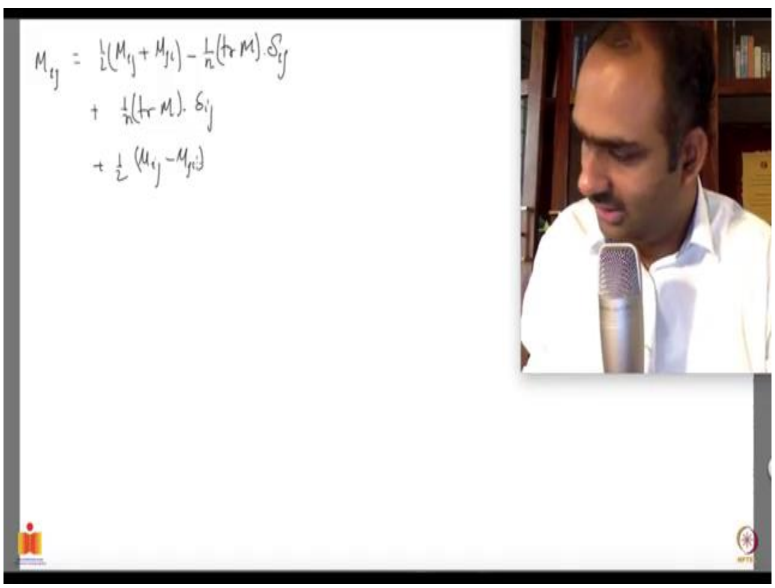
If you have a symmetric matrix then it implies that m transpose is same as im okay which means that so let me do it slightly slowly, many students get it wrong when they try they should not but they do. Let us say let us do this one. Let us say m transpose is equal to m and I want to write down this expression in component form, so I want to use indices I n j to write this.

So, what you have to do is on the both the sides, so this is a matrix equation and let us say I want to know ij th element. So, I should take the ij th element on the left-hand side I should take the ijth element on the right-hand side which is what this equation is saying and this implies and you have already seen what m transpose ij is, it is here, Mji.

So, you get Mji equals Mij that is the condition for a matrix to be symmetric. Let us look at an anti-symmetry matrix. If you look at an anti-symmetry matrix so when you transpose you get a minus sign, so you should have Mji equals minus Mij that is good.

(Refer Slide Time: 06.28)





Now that is easy let us go to the next and you can say this is a theorem ofcourse, a simple theorem. Given any square matrix m you can always decompose that matrix into a symmetric and anti-symmetric part and in fact you can do more, you can take the symmetric part and further decompose into 2 parts. One which will be traced less and another which will carry the trace.

So, I will show you a square matrix can be decomposed into a symmetric matrix, I want to as I said I can decompose the symmetric part further so into symmetric trace less matrix, matrix that will be proportional to identity which will see immediately why and then and an anti-symmetric ways, anti-symmetric, okay the proof is almost trivial.

So, let us say you have m which is n cross n, n cross n matrix so I can write down M as M plus M transpose by 2 plus minus M transpose over 2, so let us check whether it is fine? It is fine, so M transpose gets cancel by M transpose here and here you have m by 2, m over 2 and they make m, so this is fine. Okay this is your symmetric part and this is your anti-symmetric part.

Why is it symmetric? Because if you take a transpose of this entire piece you will get m transpose plus m which is same as what you already have here and if you take the transpose of this piece you will get M transpose minus M which is the negative of what you have here and that is what it is anti-symmetry.

Okay, so in the component form I can write this as Mij equals Mij plus Mji half okay that is a first symmetric piece and then you have Mij minus Mji over half over 2 that is your anti-symmetric piece. And as I said couple of minutes ago you always have identity matrix at your disposal, now identity matrix is a symmetric matrix, so out of this symmetric piece here I can take out or I can subtract any matrix which is proportional to identity and filer that out.

And what will be useful is to take out matrix which is proportional to identity ofcourse and the proportionality factor is the trace of M, so that is what we want to do, so let us have a look at it. So, let us say I here look at this, identity which is symmetric piece I multiplied with trace, trace of M. Now, this is what I want to add and subtract in the above expression.

But I have to be careful because let us say I multiply with some factor, k and I will fix the factor k that will be easier way of stating all this, so what I am saying is I will write M, so I add here k times trace of M, trace of M is some number it is not a matrix times identity because I have added this I want to also subtract it, so that I do not change the equation times identity, okay that is good.

Now, let us see what the case to I mean from here whatever you take it is all fine but what I want to do is I want to club the term with this and what I want to do is that this entire this plus this should be trace less that is what I want to do and that will fix the k for me otherwise you cannot fix k this is identically true. So, let us see, so what I want to do is I want to take half Mij plus Mji the term here and the term here plus k trace of M into identity okay I want this piece to be trace less that is what I require. Make this trace less.

So, let see, let us take the trace of it, so I take the trace of it. What do I get? I get half from here trace of the matrix and its may be and that…the transpose is the same but I hope you already know that the trace is written as M i i meaning I am summing over the repeated index here, so what is trace? Trace is the sum of diagonal entries and if you are summing over all these that is what you get the sum over all the diagonal entries.

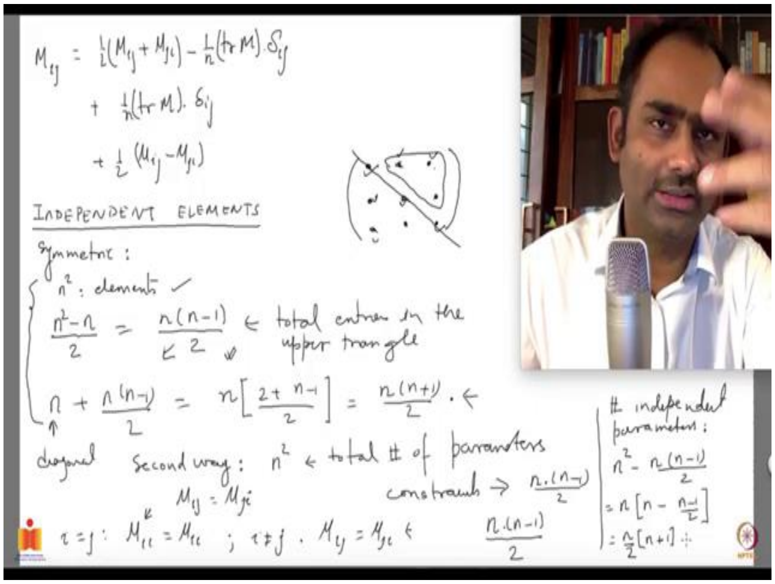
So, M11 plus M22 and so forth, so here when I am taking trace all I have to do is identify I and j and use the (())(13:04) summation convention, so it becomes half M ii which is the trace again M ii which is the trace of M plus k is a constant, trace m is a constant so there is nothing happening there and then you have identity matrix which has 1 1 1 n number of times on the diagonal.

So, if you take a trace you are adding up all those 1s which gives you n, right, because I am saying n cross n matrix. And this we want to be 0, this trace we want to be 0. Now, this is trace of m, two trace of m these two together and there is a half so which gives you trace of m, this is a trace of m here. So, from this equation I can remove the trace of m and what will be left with is 1 plus k times n equals 0.

And which means that your k is minus 1 over n. So, here in this place I should replace minus 1 over n and here again which will make it plus 1 over n here because of the minus already which is present. So, yes so that is let me write down again the result is your Mij can be written as Mij plus Mji half that is what you already have then minus 1 over n that is what you found and then you have the trace term, trace of m and then you have the identity.

And identity is delta ij, okay I am writing in component that identity I should have written delta ij here that is okay so you see minus 1 by etos n, minus 1 by n trace M times identity, these identity please write down as delta ij okay here also. Okay so that is your traceless part plus 1 by n trace of m delta ij that is your trace part so you have filtered out the trace and then you have the anti-symmetric piece half Mij minus Mji. Okay so that is the decomposition which you can always make, okay that is good and I believe you already knew it.

(Refer Slide Time: 16:16)



Now, we want to talk about, so I want to talk about the total, the number of independent entries that you have in a symmetric matrix or an anti-symmetric matrix okay that is what we want to do and I will count these in two different ways. So, first let us look at a symmetric matrix, so for symmetric matrix if you have n cross n matrix the total number of entries that you have in the entire matrix is n cross n.

So total is n square, so to begin with we have n square elements but ofcourse not all of them are independent, so let us draw here asymmetry matrix let me write down, so you have something here let us say a 3 cross 3 just for the sake of explanation, okay you have some entries at all these places.

Now, if this symmetric which means apart from the entries on the diagonal whatever you have above the same thing you have below, okay so this guy is same as that guy, this guy is same as that guy and so forth. So, if I can count how many entries are here I will be done, so how do I get this? So, in total you have n square if you count everything here, you have n square.

So, out of n square you remove what you have on the diagonal, on the diagonal you have n entries. First, second, third and so forth. So, n square n is what you have other than the diagonal entries. Now, you divide by 2 you get what you have in the upper triangle, upper half, so you divide by 2 which is same as n into n minus 1 over 2. So, these are the total entries in the upper half, upper triangle, in the upper triangle that is good.

So, how many we have in total independence ones? Ofcourse the ones which you have calculated just now these ones and I do not bother about these because they are same and then ofcourse the n on the diagonal. So, I should take these n which are coming from the diagonal ones and then these coming from here.

And how much is that? That is n, you take n outside 2 plus n minus 1 over 2, this is n 2 minus 1 is 1 which makes it n plus 1 over 2. So, these are the total number of entries which are independent for a symmetric matrix, which is correct let us check. Let us say you have a 3 cross 3 matrix, so if you had a 3 cross 3 matrix you should get 1, 2, 3 and 4, 5, 6 let us see if n is 3 this is 4, 3 plus 1 4, 4 over 2 is 2 and 3 times 2 is 6 which is correct.

Now, I want to count it in a slightly different way and arrive at the same answer. So, that was first way of looking at it. Second way not really different actually, so here I say that I have n square elements or parameters which parameterize symmetric matrix and then I say how many conditions are there, how many constraints are there on this parameters and I will remove out of the total number those constraints and I will get the number of independent parameters that parameterize the symmetric matrix.

So, that is your total number of parameters, not necessarily independent, total number of parameters, okay and how many constraints do you have? Let us look at constraints, so the constraint is that Mij is same as Mji that is what a symmetric matrix is. Now, if i equal to j it says Mii equals Mii. So, this is not a constraint, it says whatever you have is, whatever you have on the…so it says m11 is m11, m22 is m22 that is saying nothing, that is not putting any constrain, so these do not give any constraints.

And only when I is not equal to j you are going to get constraints, okay so when I is not equal to j it says Mij equals Mji which says for example m12 is m21, so that is a constraint and how many of such constraints you have? Now, you can see that I can take any of the n values then j can take any of the remaining n minus 1 values, I do not take again n because one of them is here, right when they become equal.

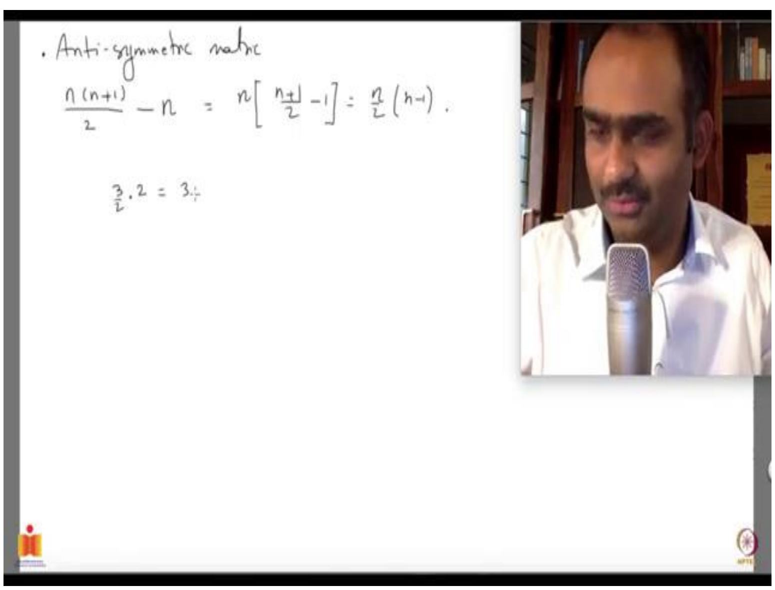
So, n can take any of the n minus 1 values which is good, so these are the number of constraints but we are still over counting because m1 to here let us say it is 1 and 2 and m21 these are same relations right, so it says m12 is m21 and again you get m21 is m12, so you counting twice, so you divide by 2. So, these are the total number of independent constraints, yes total number of independent constraints.

So, here you have the total number of constraints is n into n minus 1 over 2, okay. So, how do you get the total number of parameters, independent parameters? You get by taking the total number of parameters that you have minus the total number of constraint equations that you have which is n into n minus 1 over 2.

Let us see what that it is, this is n I have taken out common and you have n minus n plus n minus n plus sorry that is what am I doing? Sorry, okay so I have taken n outside so you have n and this n is gone, so only minus you have n minus 1 by 2. What am I doing? Yes sorry, n minus 1 by 2 that is correct, so you have n, 2n minus n gives you n that makes a plus 1 and you have a half here which is let us see what you got earlier n into n plus 1 by 2 which is same as before.

So, that is how you count for a symmetric matrix that it has these many independent parameters and for anti-symmetric matrix it is the same calculation except for the fact that on the diagonal you have nothing, the diagonal entries are 0, so you remove those n entries.

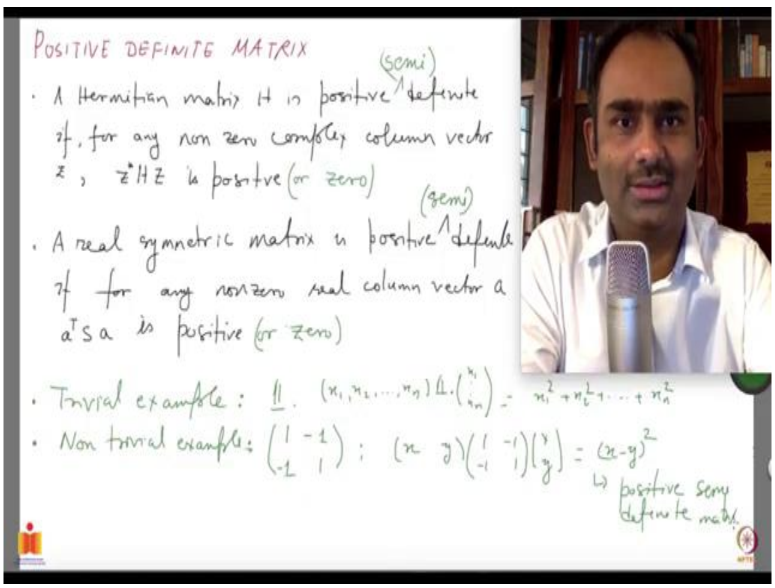
(Refer Slide Time: 25:21)



So, let us see, so for an anti-symmetric matrix what we can do is we can take the previous result which was n into n plus 1 over 2. N into n plus 1 over 2 but now I have nothing on the diagonal they are all 0, so there is nothing to choose they are not any you cannot make a choice of what should be or the first entry, what should be the second entry on the diagonal, so you remove them, okay because that is in here. So, what do you get? You take n outside you have n plus 1 over 2 minus 1which is n over 2 n plus 1 minus 2 which is n minus 1.

So, n into n minus 1 over 2 which is the correct result. And let us check for a 3 cross 3 matrix you should get only 3, what you have on the upper corner, okay so you put 3, n equal to 3 so we get 3 by 2 into 3 minus 1 is 2, so it is correct. Okay that is fine, now I want to define what is a positive definite matrix maybe I should go to the next page.

(Refer Slide Time: 27:40)



Okay, so I will write down Hermitian matrix, you know Hermitian matrix is if you take the dagger of it you get it back, so H dagger is H. And dagger involves doing a complex conjugation and the transpose both. So, here is a definition, shall I use a color? Now, let us see, color okay, so positive definite matrix and I will also look at positive semi-definite matrix.

So, anyway let me define positive definite matrix first, go back to back, so the definition is here, Hermitian matrix H is positive definite, okay if for any non-zero complex column vector that you take and if you sandwich on the left hand side on the right hand side of H these column vectors you should get a positive quantity, so let me more precise here.

If for any non-zero complex column vector Z, vector Z H star, so H star is the complex conjugate of Z sorry not Z, Z star is the complex conjugate of Z, H Z if you calculate this okay, so here is a matrix H you on the right hand side you put a column matrix, on the left hand side you have put a row matrix after taking the conjugate and I am saying that no matter what z you take, for all z if this quantity turns out to be positive then I will say that h is a positive definite matrix.

So, let me read Hermitian matrix H is positive definite for any non-zero complex column vector Z, H star HZ is positive, okay that is the definition of it. Another definition I will use some color here and the definition is a Hermitian matrix H is positive semi-definite so I put semi here, if for any non-zero complex column vector Z H star H is positive or 0.

Okay so the thing which I have written in green applies for only semi-definite. So, if you always get positive not zero then it is positive definite if you can also get 0 then it is called positive semi-definite. So, that is the definition of it will talk a little bit about this in a while but before that I will say something about a symmetric matrix.

So, you know a symmetric matrix are the brothers of or cousins a Hermitian matrix, so what is a Hermitian matrix in the complex world is a real matrix in a real world, world of real vectors. So, symmetric matrix is positive definite is it gives you a positive result for whatever z you take. So, you put a z let us say the symmetric matrix is what I am calling S, and you multiply by any column which is a real column on the right-hand side and with the same column written as a row on the left-hand side.

And if you get a positive value for it then it is a positive definite matrix, so let me write it down here. Real symmetric matrix is positive definite, is definite if for any non-zero real column let us call that column as A you get A transpose s a, a as positive, okay if this is true then your matrix s is positive definite.

If as before you get 0 also, so let us say you can get 0 for certain vectors then you call it is positive semi-definite or 0 let me out it in brackets. So, these are the definitions. Now, first let me give you a simple example I think that will be (())(34:01) thing to do, yes, first let me give you a trivial example, so I am now looking at some the trivial is symmetric matrix is the identity matrix and this is positive definite.

Because if you take any non-zero vector let us say x1, x2 and so forth xn and you have identity matrix here and you have this column x1 to xn what do you get is x1 square plus x2 square so and so forth xn square. Now, no matter what x you take, if x is…x1 is minus 5, x2 is minus 20 some of them are 0 and some of them are positive no matter what because the squares are involved here whatever you get is always positive.

The only way you can get a 0 here is only if you take the vector itself to be 0, if all these x are 0 then you get but then we are saying the definition was only non-zero vectors. So, this is clearly a positive definite matrix. Let me give you a non-trivial example. Please pay attention to the positive definite matrices because we are going to utilize this later when we talk about small oscillations.

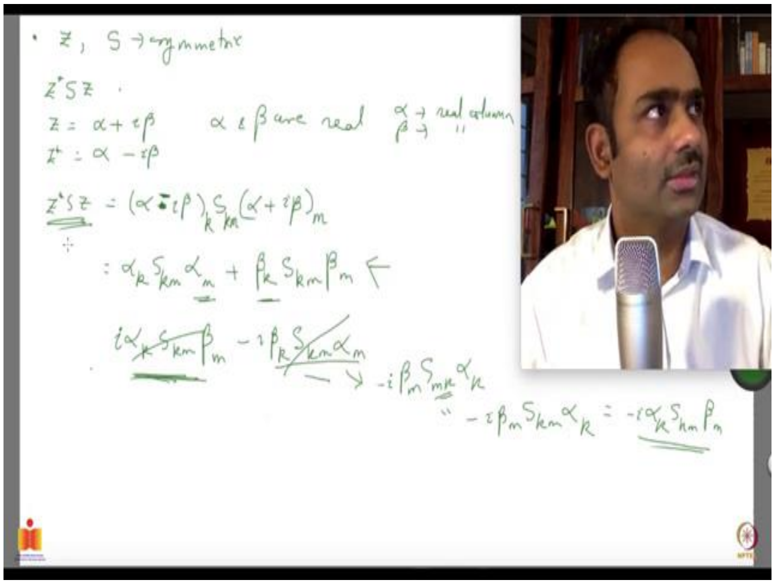
Okay, so let us see here, so I take the matrix to be 1 minus 1 minus 1 and 1. Let us see now what we get, so what you do is let sandwich x y or you could have written x1, x2 does not matter 1 minus 1 minus 1 1 and x y. So, I am taking any x, any y and what do you get here? You get here is x minus y square. So, can you tell me what kind of matrix this is, is it positive definite matrix, is it positive semi-definite matrix or none of those?

See not every matrix will be positive definite or positive semi-definite they may be neither of this, so what do you think is the case here? So, clearly no matter what x and y you choose you are going to get, you are going to never get something which is negative because of the square here. But you can get a 0 because if x and y are same you get a 0.

So, if you choose 1 1 for the vector you get a 0, you get you choose 2.5, 2.5 you get a 0 but if you choose x and y to be different you get non-zero. So, this is clearly an example of a positive semi-definite matrix. Okay that is good, now you may ask what happens if I take a real symmetric matrix which is let us say positive definite or positive semi-definite and I sandwich between two, put form the left or z star and from the right is z. And where z is a complex vector, what happens to that?

Do I get something complex? Do I get something real? Do I, can I say something about it being positive definite or not in the complex space and that is what I want to show you now and the algebra is again quite simple.

(Refer Slide Time: 38:36)



So, let us say I take a complex vector z column vector or yes column vector and let us say I am given a positive definite matrix, forget positive definite, let us say a symmetric matrix s, okay will not specify any property of s, so I just take s which is just symmetric. I am not assuming any other property right now and let us look at the quantity z star s z and ask what this is?

Okay so first thing I do is I take the z and write it in terms of real quantity, so let us say the column z can be written as this where alpha and beta are real. And ofcourse alpha is a column and beta is also a column, real column vector and this is also real column vector. So, z star would be your alpha minus I beta and let us now construct z star as z.

So, we have an alpha plus i beta times yours s times your alpha sorry there was a minus sign here, I put a minus sign here, alpha plus I beta. Now, if you want to yeah I want to write this out in component, so I say what am I using k. So, let me put a index k here, m here and m am also using summation convention, okay so there is a sum over k and m and this is how you write a product okay that is good.

So, I have 4 terms here, so I multiply by alpha s alpha so I get alpha k s k m and alpha m let me multiply the betas you get I square is minus 1 and there is minus already there so that makes a plus, so we have a beta k s k m beta m that is good. Then you have the cross terms involving alphas and betas, so let us look at them.

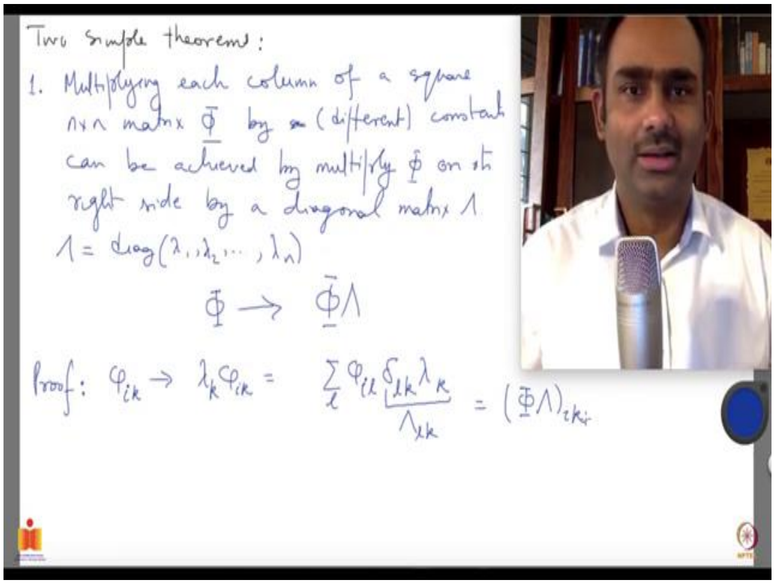
You get alpha k s k m beta m and there is a i, I mean I here and this one will give you minus I beta k s k m and from here you get an alpha m. Let us look at these two terms. You see these k and m are repeated they are dummy, so I can interchange, so instead of k I can start writing m and instead of m I start writing k, so I look at this term, this one I will write it as I keep the minus I also with me, minus I beta m s m k alpha k.

Okay so I have just interchange the indices, so which I am allowed to. Now, your s is a symmetric matrix so s m k is same as s k m as you saw earlier, so I can write this as minus i beta m s k m alpha k which is nothing but minus i let me write down first alpha k skm beta m this term is same as this term except for the minus sign.

So, these two cancels, so this cancels against this and what I am left with is only these two and where alpha and beta are real vectors. So, now you see and first thing you realize is that even though our z is complex you can always split this into terms involving only the real parts and there is no complex left around. And also if s is positive definite then this will be positive and this will be also positive, both these will be positive.

Which means this will also be positive even in the complex space, so that is one thing which I wanted to tell about which we will utilize later when we talk about oscillations. And I want to talk about two very simple theorems which will again be very useful, let us talk about them. So, I call them two simple theorems because they are simple, what happened?

(Refer Slide Time: 44:21)



Two simple theorems, theorem number 1, may be some color will be nice, let me see what I have written, so let me write down. So, the thing which I wanted to tell you here is the following. Let us say you are given some matrix, let us call it phi, okay and I want to multiply some constant in each column of the matrix, so you look at the matrix phi as a collection of columns, so column number 1, column number 2, column number 3 and so forth.

And each column I want to multiply some number, so in the first column let us say I want to multiply 2, the second column I want to multiply 3, in third columns I want to multiply 25 okay like this. So, what should I do to get this and similarly given a matrix phi i want to multiply certain numbers in each row, so first row I want to multiply something, second row something and so forth.

So, that is what I am looking at, let me write it down. Multiply, multiplying each column of a square n cross n matrix phi by a constant and because this constants could be different let me write down this by constants can be achieved by multiplying your phi from right hand side, on right hand side not from, on right hand…on its right side, on its right side by a diagonal matrix lambda and what is that lambda, lambda contains all the numbers which you want to multiply.

So, lambda is diagonal matrix with entries lambda 1, lambda 2 and so forth lambda n, okay these are the lambda 1, lambda 2 these are the things which you want to multiply in each column and what is the result, can so here I am writing multiplying each column of a square n cross n matrix phi by constants can be achieved by multiplying phi on each right side by a diagonal matrix lambda, where lambda is this.

And how do you achieve it? By multiplying on the right hand side which is your phi if you do this then you will achieve what you have desire. Let me give a quick proof, its trivial, so what I want to do is I want to start from let me use small phi to denote the entries, so let us say I have phi I k, okay k is the one which labels the columns.

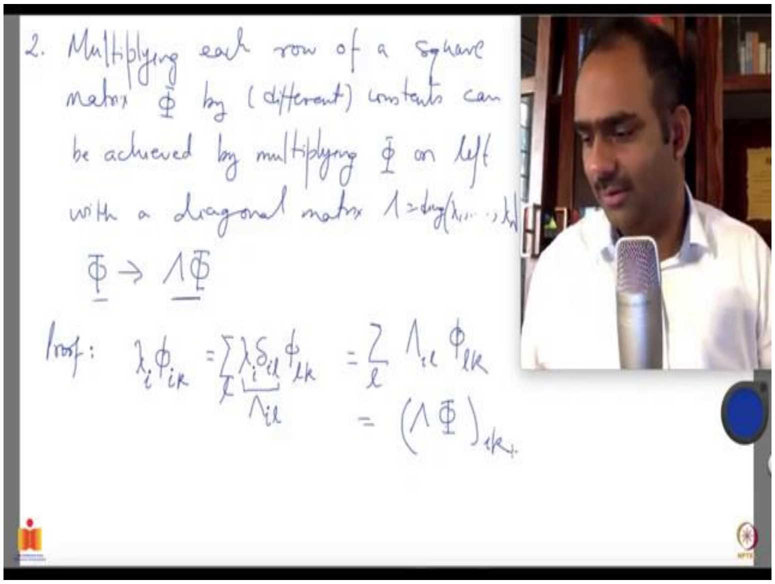
So, what I want to do is go from here to phi i k but each entry I mean, each column gets multiplied by some k, so lambda k, right, k, remember k labels the column so that is why I am multiplying by lambda k, that is good, that is what I want to achieve and what is this? This right-hand side I can write as, so there is no summation over k here, no summation over k even though the k is repeated that should be cleared.

Now, I can write this phi I k as phi i L delta kl or let me write lk it is symmetric anyway so I can write delta lk and you already have a lambda k sitting with you, okay that lambda k is there. So, phi i l delta lk makes phi ik there is a summation over l which is there but not over k, let me make it explicit by putting summation over, okay.

So, you agree with this? Now, this I define as some matrix, capital lambda and lk remember there is no summation over k so that is why I am able to put this two indices, see this is this has two free indices l and k and that is why I can create this matrix capital lambda with two free indices l and k. So, that is correct and what is phi Il times lambda lk? This is just the rule for a matrix multiplication, so I am summing over l.

So, this I write as phi times lambda and we are here looking at ik th element, okay so that is good if you want to multiply each column that is what you should do. So, put create a matrix lambda which is diagonal and put all the numbers which you want to multiply on the diagonal entries and if you construct this quantity that is what is going to give you what you desire.

(Refer Slide Time: 51:18)



Okay theorem number 2, should be obvious now, multiplying each row of a square matrix phi n cross n by again I put different in brackets constants can be achieved by multiplying phi from the left, by multiplying phi on left with again the same diagonal matrix which we had earlier, matrix lambda which is diagonal of, whose diagonal entries are the constants which you want to multiply, okay that is what it is.

So, phi should go to lambda phi that is what it is, let me prove it again it is simple, proof as before. So, you have here this thing, so take the phi ik element now I labels the row, so because you want to multiply in each row something, so ith row you want to multiply lambda I, so this is what you want to achieve, this is where you want to arrive at.

And this you can write as before as phi again there is no summation over I that you remember even though they are repeated, so let me write phi lk delta il, if I write this I have made this into phi ik, so that is how I have lifted off I from the phi and put on the delta, this you can always do, so given a one index you can always pull out from some quantity and put on another using a delta function.

That is fine but our lambda i is still there, there is no summation over i but there is a summation over l let me make it explicit by putting this, that is good. Now, again as before I define this as matrix lambda and i is a free index and l is also a free at least in here, in this part and you see now what you have is summation over l lambda il phi lk which is nothing but the matrix multiplication rule lambda times phi and you are looking at the ik th element of it. Okay so we see that we have achieved it, okay I will stop here and we will talk more about matrices and other related quantities which we are going to utilize in the later parts of the course, okay. So, see you in the next video.