Introduction to Classical Mechanics
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Lecture 14
Momentum Conservation

In the last video I talked about symmetry under time translation and today we will talk about symmetry under space translation. Imagine if you have a system which is, I mean imagine a system that is isolated from everything else, let us say they are several particles in that system which are interacting with each other and that system is not influenced by anything else, then if you take that system as a whole and displace it to some another location everything that is happening within the system will remain unaffected.

So you will not be able to say whether the system was here or here when you read the experiment, so that is the symmetry which we are talking about. And what I will do is I will start by a simple example to appreciate what we trying to say and the simple example is that of two particles which are interacting with each other and I imagine that both particles are only in let us say can live only in the X-axis, only along the X-axis.

So the system lives only in 1-dimension, I mean 1-dimensioanl space this what I mean, so let us have a look at this one.

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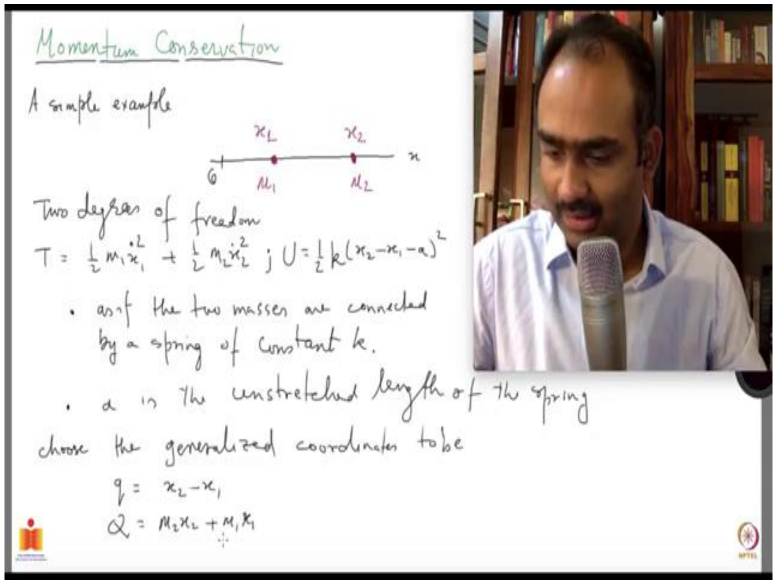

So that is your let us say X-axis some origin O and your (somehow) feels nice if you have color let us see. You have a particle here and another particle here, let us say the coordinates are x1 and x2 and their masses are m1 and m2. So how many degrees of freedom does this system have? What do you think? Meanwhile I will choose make this black in color.

So this system clearly has 2 degrees of freedom, 2 particles each allowed to move only in along the X-axis. So there are 2 degrees of freedom, that is good. Let us write down the kinetic energy and the potential energy to construct the Lagrangian. So your t is half m1 x1 dot square plus half m2 x2 dot square and I choose a special kind of a potential for this problem I say that the potential is of the following form, let us see what it is.

So half k x2 minus x1 minus a square, so what I am imagining is that these two particles are let us say connected by a spring of constant k and the (what you said) the unstressed length of the spring is E. So the spring length is A there is no force is on either of the particles, the moment you stretch or compress, meaning x2 minus x1 becomes different from A then there is a force and that is the spring constant.

Now this may look like a simple example with 2 masses and a spring but you will see later or maybe you already know that this is a very good approximation for real system, so you will encounter many system, many systems in which their constituents will to first degree of approximation in interacting in such a manner. If you do not know this already will have later occasions to talk about this but nevertheless we I mean anyhow we are going to choose such a potential for our system.

Let me write down here, let me just write down whatever I have said here. So we are imagining as if the two masses are connected by a spring of constant k, I hope that is clear from here, that is x2 minus x1 is the suppression between these two masses and if you subtract A form it, let me write down what A is first then I will, A is the upstretched length of the spring.

So if x2 minus x1 is different from A it means that either it is compressed or stretched, so there is a force. Now you see there are no constraints on this system, so the Cartesian coordinates are good you can use them as your coordinates for writing down Euler Lagrange equation so there is no problem with that. But I will use a different set of coordinates other than the Cartesian to demonstrate the power of using some good coordinates that is what I want to do.

So I will choose instead of the Cartesian coordinates which as I said are perfectly good because of them being independent of each other, let me choose the following. Let me chose the following to be generalized, chose the generalized coordinates to be following. So one coordinate I call it small q which will be x2 minus x1 and another coordinate I call capital Q which I will call m2 x2 plus m1 x1 I choose them because I can do so.

Let us say if you do not want to choose it, you want to choose the generalized coordinates to be x2 minus x1 and another one to be x2 plus x1 you can do so, there is no problem with that. But I want to show why this choice is really fantastic choice and you will see soon why, let me see, okay maybe I should go to the next page.

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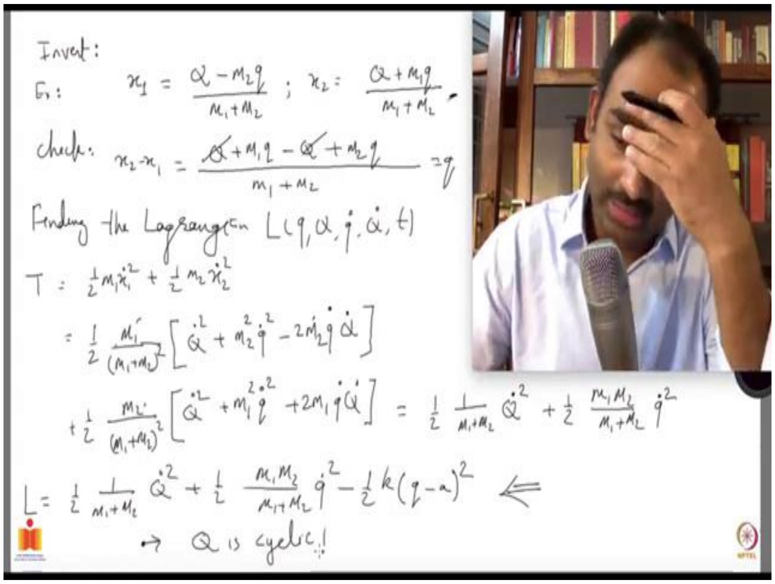

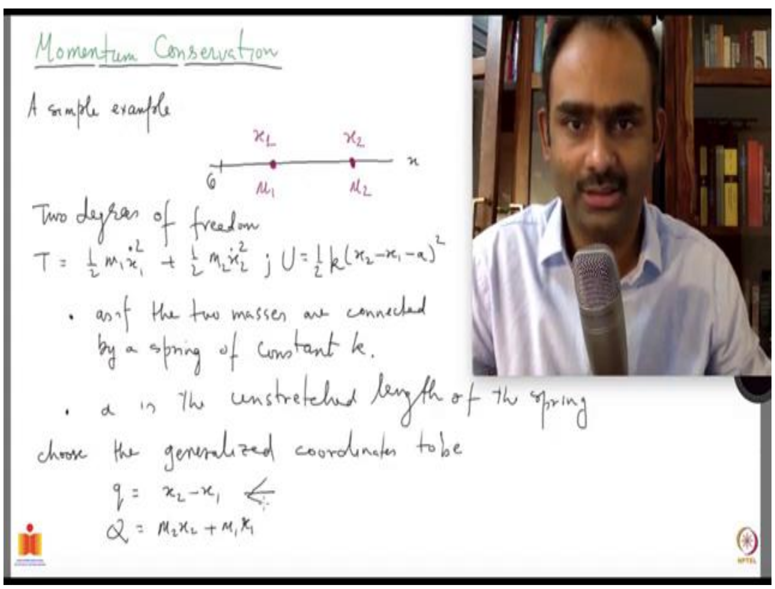

So take those coordinates and invert them now, so determine x1 and x2 in terms of small q and capital Q that is what you have to do and you should do the following exercise and check that indeed you get this result. So you get x1 as capital Q minus m2 small q over m1 plus m2 and your x2 turns out to be Q, capital Q plus small m1 (sorry) capital Q plus m1 small q over m1 plus m2.

So that should be easy let us check whether this is correct, so you should do this, it is minor trivial algebra, nevertheless do it. Let us check whether this is fine. Okay after always do a check after you have done some algebra and let us see what that is. Let us check whether x2 minus x1 comes out to be q, okay that is what should happen.

So I calculate x2 minus x1 so both have m1 plus m2 in the denominator and 1 plus m2 and let us see what is the numerator, x2 is this Q plus m1 q, m1 small q minus x1 so minus small minus capital Q plus m2 small q, this capital Q cancels and you have m1 plus m2 you can take out and q will be the factor and which you see is q and that is what we started with.

So let us go back here, okay, so indeed what I have given as result is correct, now our goal is to find first the kinetic term, potential term main term of the generalized coordinates, okay so that is what will do. So you take t find in the Lagrangian and oops something I have done, how come, let us see. Should be able to get rid of it, it is not bothering us anyway so yeah I can cancel here.

So what yeah, okay there is some Lagrangian in finding the Lagrangian L oops okay so something is wrong I think I know let us go to tool bar yes, now it is fine. So L q Q possible t let us see whether there is a t here and ofcourse should have written down q dot, capital Q dot and then t that is what we are doing right now.

I can remove this tool bar, good. So let us find out T now, so your T is as before half m1 x1 dot square plus half m2 x2 dot square and let us substitute our x1 and x2 from here. So what do we get half m1 x1 dot square this will involve m1 plus m2 square in the denominator so let me write that down m1 plus m2 whole square and then your x1 dot square will have this piece.

It will have q dot square plus m2 square q dot square minus 2 m2 q dot capital Q dot. So here you see that you have off-diagonal terms also. See this one is q dot q dot so it is a diagonal term q dot, small q dot small q dot that is a diagonal term but small q dot with a capital Q dot that is an off-diagonal term.

Let us write down another, the next term half m2 x2 dot square so I should look at this one now. Again you have m1 plus m2 whole thing squared in the denominator and then you have q dot square capital Q dot square plus m1 square q small q dot square plus 2 m1 small q dot capital Q dot. Let us see what we get when we add the two.

You see that these crossed terms, off-diagonal terms so this will have m1, m2 this will have m2 m1 which is same as before and then you have small q dot capital Q dot which is also here, factor of 2 and a factor of half which is also present here and they are opposite in signs, so these two cancel. So there you go away and what you are left with is the following.

Let me just write it down, what you are left is this, left with is this half 1 over m1 plus m2 capital Q dot square plus half m1 m2 in the numerator and 1 plus m2 in the denominator and small q dot square that is what you are left with all off-diagonal pieces are gone that is good. So let me write down the Lagrangian directly because u is simple so my Lagrangian is half I am writing that down again anyway let me write down.

Q dot square plus half m1 m2 over m1 plus m2 q dot square minus u and what is u? U is half k it has x2 minus x1 which is q, small q okay, q minus A, A is a constant remember that square, so that is what your Lagrangian is. Ofcourse I can redefine my small q and absorb that constant A which is sitting in the last term but I am not really bothered with that one so I will leave it.

Now, what is so nice about this Lagrangian or what is so nice about the choice of coordinates that we have made? Can you spot something? You see the capital Q does not appear in this Lagrangian meaning Q, this capital Q is a cyclic coordinate, so that is what so nice about the choice I have made. Q is cyclic okay I hope you appreciate this

You could have (there is a phone call just again) you could have worked with Cartesian coordinates which was fine, they were independent but then neither of them, neither x1 nor x2 is a cyclic coordinate okay. You could have worked with the difference of the x1 and x2 and the sum of x1 and x2 as the generalized coordinates and if you do that exercise you will find that you do not find any cyclic coordinate there as well.

But if you make the choice which I have made now you will find that one of the coordinates is going to turn out to be cyclic which is the case here. Let us go that is very good, that is what is good about our choice of coordinate. Anyhow so you already know that if a coordinate cyclic then the corresponding generalized momentum is conserved which means that it will go to the next slide.

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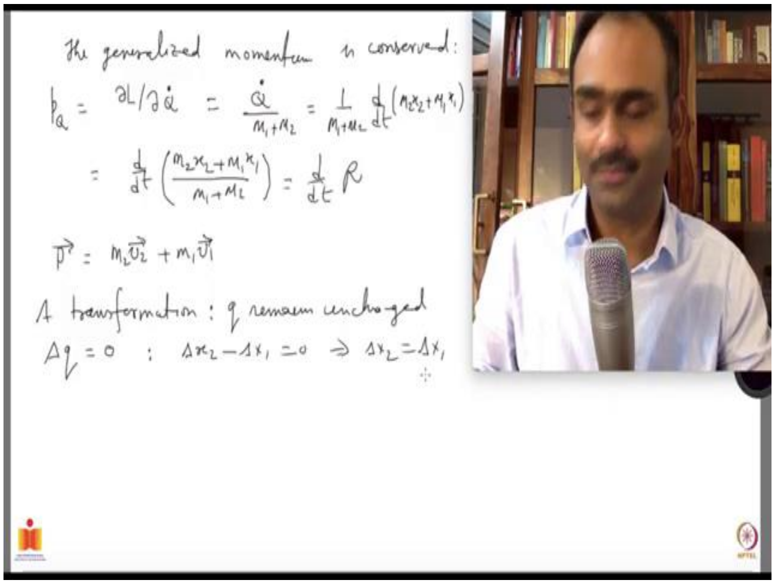

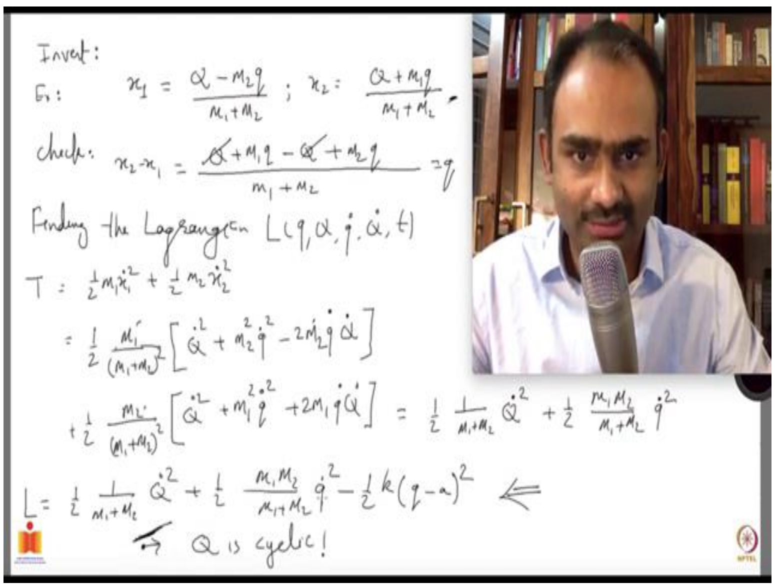

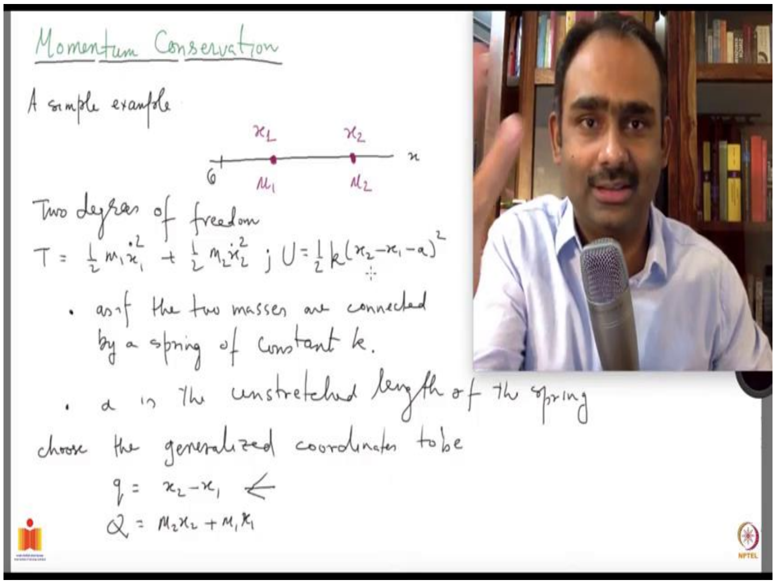

Which means that the generalized momentum corresponding to Q is conserved just as second, please stay back, is conserved that is good. So let us find out what that thing is okay so the generalized momentum p q is del L over del q dot remember this and what is del L over del q dot? Let us see, this is the only place where q dot appears there is no other place and if you differentiate you will get q dot over m1 plus m2.

So you get this is equal to Q dot over m1 plus m2 which is same as what is Q dot? Let me write down 1 plus m1 over m2 and what is Q dot? Q dot is d over dt of q, and what is q? q2 is x2 sorry m2 x1 sorry m2 x2 plus m1 x1 that is good which is d over dt of m2 x2 plus m1 x1 over m1 plus m2, does this appear familiar?

This r, or the quantity in the round brackets is the location of your centre of mass, okay the centre of mass of your system is located at this place. So all you are getting is that the velocity with which the centre of mass is going to move is a constant, that is what it says which is true, and just a second let me see what else I wanted to say.

So you see you have got this result because of the symmetry that was present in the system that if you take the entire system and move rigidly to some other place the Lagrangian would not change I hope you can see that thing here. The first place when we wrote down the Lagrangian here. See the you would cares not about x2 and x1 individually does not care where exactly x2 where exactly the second particle is or first particle is, all it care is about is the difference between those locations.

And here anyway there are velocities so it does not care about what the locations are and that is why you are getting this conservation law that the centre of mass is moving with the constant speed which is also equivalent to saying that your total momentum of your system is conserved. So if you look at this numerator, the numerator is just m2 v2 which is the momentum of particle number 2, m1 v1 which is the momentum of particle number 1 and that is the total momentum of your system which is conserved.

Because the total time derivative of this is 0 and this conservation law you have got because of the symmetry which we have in the system that is one but also we have realized that a good choice of generalized coordinates will make this thing apparent and you will be able to identify or you will be able to make some of the coordinates cyclic depending on what the situation is.

Let me say a little bit more about the coordinates q and capital Q. Imagine that you take your system which has these two particles okay and this guy is here that guy is here which means the small q and capital Q have some values. Now let us say I do something to the system I push this here and pull that one there, so you have new x1 and x2 values which means you have new q and new capital, small q and capital Q values, that is what will happen if you change the coordinates.

Now, because small q and capital Q are independent I can chose not to change the small q and change the capital Q. So that is the kind of transformation that I want to look at. So let us look at a transformation where your q remains unchanged a small q unchanged and only capital Q changes and I can do so because small q and capital Q are independent. If they were not then I could not have done this.

So let us see what I am saying is I do some, I move the two particles at different locations such that the change in q, small q is 0 which means x2 minus x1 does not change which means delta x2 minus delta x1 is 0 or delta x2 is same as delta x1. And your capital Q has changed, now what does it mean that small delta x2 is same as small delta x1? Meaning the change in the location of particle number 1 equals in the magnitude and direction both to the change in the location of particle number 2.

Which means that I have taken entire system and moved both the particles by equal amounts in the same direction either this way or that way. So as if I have taken the entire system and rigidly moved it from here to another location that is what it means and you see when I do so, when I do so it does not matter what delta what capital Q is, if I have chosen to keep small q to be 0 then no matter what I do to capital Q my Lagrangian does not change because that does not appear in the Lagrangian.

And you should check that capital Q just determines the anyway we have seen it, just determines the location of the centre mass, it was 2 x2 plus m1 x1 all that is missing is the denominator with total mass which you can multiply. So that is one thing which I wanted to say. Now if you have an isolated system and with lots of particles in them and you have a symmetry in the system which means that you can take the entire system here or this way or that way, things should not change.

Then you should be able to choose your generalized coordinates in such a manner that you will get 3 cyclic coordinates and each will correspond to a translation of the centre of mass either in the x direction or y direction or the z direction.

That is good so we have talked already about the conservation of energy that arises from translation symmetry then we have talked about conservation of momentum, the total momentum of the entire system through this example which arises because of space symmetry and you can imagine that if we take a system which is again isolated from everything else and I rotate it about some axis then nothing should change.

The system should not know that whether it is oriented like this or that and we should have a corresponding conservation law arising from such a symmetry and they should be corresponding generalized coordinate which should be cyclic. So that we will take up it next video and yes that is all what I have to say in this one. See you then.