Introduction to Classical Mechanics
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Lecture 13
Energy Function, Jacobi's Integral

In the last video we were looking at the symmetry under time translation and we defined what is called Jacobi's Integral, which is also known as energy function or Hamaltonian. So we will talk little bit more about that and will take two simple examples in today's video for appreciating these quantities better.

And in the next one we will talk about in the next video and or so will start talking about symmetries under space translation. So let us start by recollecting what we talked about last time and we will take two simple examples after that.

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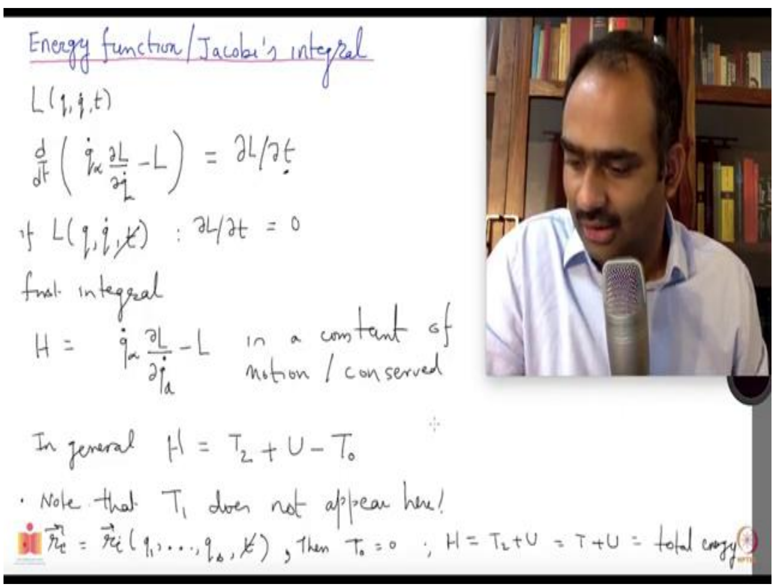

So as you may recall, in the last video we saw that if you have a system that is described by a Lagrangian L, q, q dot t this, so right now I am keeping the time dependence to there so there may be an explicit time dependence. Then we saw that if I use equations of motion that is Euler Lagrange equations of motion then I can show that the total time derivative of the following quantity.

So d over dt q alpha dot del l over del q alpha dot minus L recall that there is a summation implied over alpha because of the instantaneous summation convention and we showed that this total time derivative is del L del t. Now, if L is not a function the Lagrangian is not a function of t explicitly, if this is true which just means the delta L over delta t is 0, then the first integral then the first integral of motion H q alpha dot del L over del q alpha dot minus L is a constant of motion because this guy is 0 and the total time derivative of this is 0 and this what is what I am writing is H is a constant of motion or it is conserved.

Very good, and we also found that in general, your in general H is T2 plus internal energy minus T0 and recall T2 is the term which is quadratic in generalized velocity and T0 is the term which is of or let us say independent of the generalized velocities and same is U, so U also does not depend on generalized velocities in this case so U and T naught are at the same footing here.

And that is what we found, and note that T1 does not appear in this expression, let me write it down. Note that, T1 does not appear here and we also saw that if the transformation from the Cartesian coordinates to the generalized coordinates does not involve time explicitly, meaning if let me complete the sentence, if the transformation does not involve time explicitly going from the Cartesian coordinates to the generalized coordinates, then we could show that the H, the Jacobi's integral is same as the total energy of that system, that is what we saw.

Let me write it down, so if your r, i these are your Cartesian coordinates and this is your transformation rule which takes you from the Cartesian to the generalized coordinates. Let us say the arise degrees of freedom. And if this does not explicitly depend on time so I am removing t then all the terms which are linear or degree 0 in generalized velocities they disappear.

So for example T0 is 0, T1 is also 0 and then your H is just T2 plus U but T2 is the only term which appears in the kinetic term so which is same as T and this is the total energy, that is what we talked about last time and from here we want to take it further or may be let me leave it like this.

Nevertheless it is possible that your transformation from Cartesian coordinates here to the generalized coordinates involves time explicitly even then it is possible that the Lagrangian does not dependent on time explicitly. And clearly in that situation your dh or dt is this quantity, this will be 0 because Lagrangian will not dependent on time explicitly. So H will be conserved.

But because we are saying that there is an explicit time dependence then your H will not be the total energy, because then H will have the form T2 plus u minus T0. So that is remark which I wanted to make may be I will write it down so that we have it neatly written here, so let me write down.

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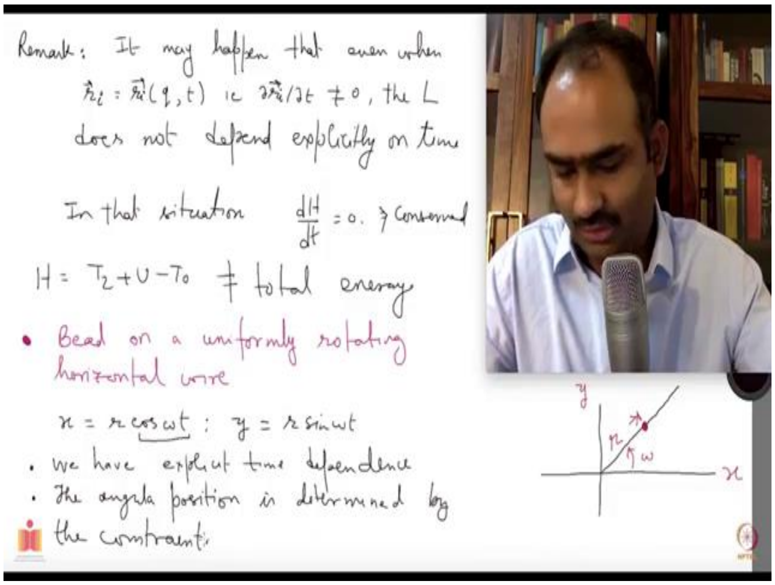

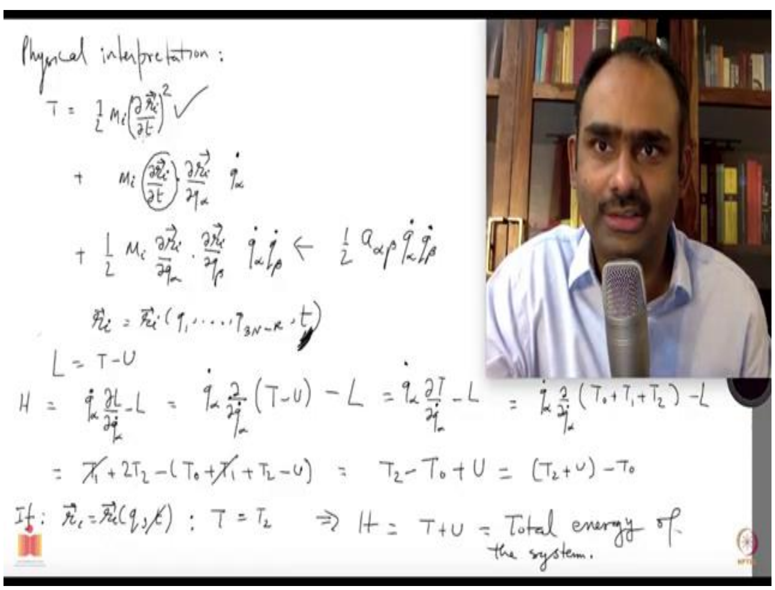

Remark, it may happen that even when your transformation from the Cartesian to generalized coordinates is a function of time explicitly, let me just write q that is del ri over del T is not equal to 0 it is the same thing. The Lagrangian does not dependent explicitly on time that is nice which means what if that happens I will give you later an example of this but let us say we believe that can happen then in that situation, let us go back, what is true in that situation?

Here, there is it, here, see this is H this piece and this is del over del t I am saying del L over del t 0 so dh over dt will be 0, dh over dt equal to 0, meaning it is conserved, H is conserved but your H would be T2 plus u minus T0 because all these terms I mean not all this but T0 appears because of the presence of time in this transformation law, let us see, I think we can see here, yeah here, you see?

All these terms were generated because of explicit time dependence, this one here and this was anyway the one, only one which is independent of time derivatives with respect I mean time derivatives of r. So you see in that case this is not equal to total energy. So which just means that your Jacobi's integral whether it is your H is conserved quantity or not it is a different thing, a different question from asking whether your Jacobi's integral is equal to the total energy.

So these are two different questions as you have seen here just now. So let us take simple example, I will take two simple, very simple examples to underscore these points. So example number 1, let me use some nice color if I can, let us take, okay let us anyway this is some funny color but I will take it.

Bead on a uniformly rotating wire that is lie in a horizontal plane okay, bead on a uniformly rotating horizontal z o n t a l (horizontal wire) creating horizontal wire, perfect. So the situation is this, so let us say you have this wire which goes to infinity and you have a bead here. Bead is like “mala mai moti”, okay that is the thing.

So it can slide along this wire, so imagine a thin wire not of this thick pen and this thing is moving about a fix point uniformly. Uniform means the angular velocity is not changing with time that is what I mean, so this is moving and there is a bead which will which can slide (across) along the wire that can happen.

So what we are asking is, what happens to that bead when this thing is moving rotating uniformly, and the plane is horizontal so I am not talking about such a rotation, I am talking about a horizontal plane. So gravity has no role here, good so what happens if this is the case, here. So here let me draw the diagram, so that is your wire it goes along this I am just feeling like putting more color today.

Let us see if I can put, so here is a bead and this thing is moving with some angular velocity omega. That is your x axis, that is your y axis, perfect and the distance from here to there that is r, nice, looks good. So well clearly I can write down x equals r cos omega t and y is r sin omega t, so your system is clearly 1-dimansional there is only one, so instead of using the two Cartesian coordinates x and y I am using the Cartesian the polar coordinates r and theta.

But theta is already determined by the wire itself, so wire fixes what the theta would be for the particle, so there is only one coordinate left that is r and you also notice that there is an explicit time dependence since the transformation from r to q where q is r. You see here there is a factor cos omega t, so time is explicitly here and ofcourse there is an implicit time dependence here r of t.

So we have an explicit time dependence, let me write here, so we have explicit time dependence in the transformation from r to q and then also as I said the angular position is completely determined by the wire, so the constraint that this wire is moving at angular velocity omega that determines where the bead would be as far as the angle is concern.

The angular position determines the, sorry the angular position is determined by the constraint, that is good. I believe we have already done this example earlier, so you can do the following.

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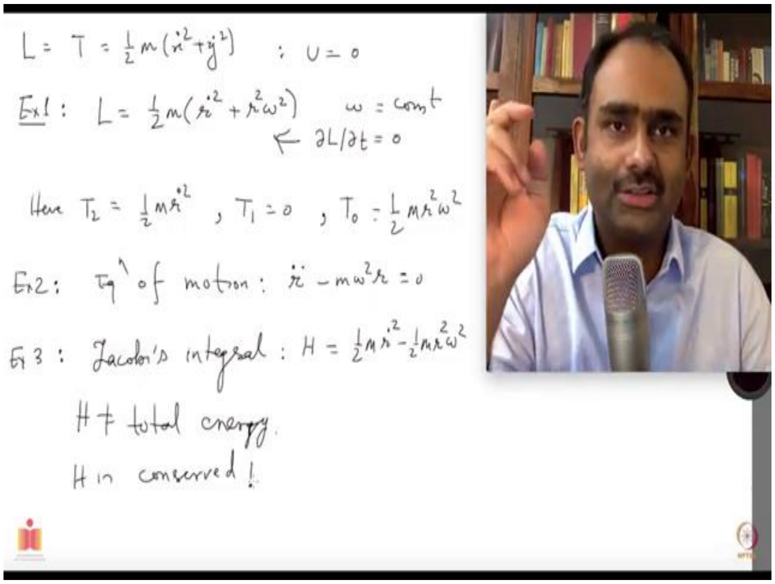

So clearly the Lagrangian in this case is just the kinetic energy because there is no potential energy involved in this problem. So it is just the constraint, there are no other forces which are acting on the bead other than the force of constraint.

So your kinetic energy is all you have in the Lagrangian, so half m x dot square plus y dot square and as I said the potential energy is 0. So exercise number 1, please do it. Show that your Lagrangian in the generalized coordinates which is r now is half m r dot square plus r square omega square, where omega is a constant. We said that the wire is moving uniformly, so omega is constant does not dependent on time.

Now, when you show this you realize that the Lagrangian does not depend on time explicitly, so clearly this L does not depend on time explicitly which means the partial derivative of L t is 0 even though your the transformation from r to q does involve time explicitly, so that is the example of such a case.

Now let me write down here, so here what is T2? T2 is half m r dot square, what is T1? 0, because there is no term which is linear in generalized velocities. What is T0? That is half m r square, omega square, very nice.

Next exercise for you find out the equation of motion for the system. So you please find it out and the equation of motion use your Lagrange equation and get the following. Equation of motion should be r double dot minus m omega square r equal 0. Now exercise number 3 it is also simple, find out the Jacobi's integral that is find out H and you know the formula of H.

So show that you get the following, H equals half m r dot square minus half m r square omega square. That is your Jacobi's integral in this case, is it equal to the total energy of the system? Does it equal the total energy of the system, what do you think? And it is H conserved, so that is the two thing, these are the two things, H is not the total energy of the system because your total energy was purely kinetic which was here, this is the total kinetic energy.

So your half m r dot square which is here but then you have a plus half m r square omega square and here you have minus half m r square omega square. So clearly this is not the total energy, so we see that H is not equal to total energy of the system, not surprising because your as we talked earlier transformation loss do contain time explicitly and is H conserved? Yes, H is conserved.

That is our example number 1 and I hope you appreciate more whatever we talked about Jacobi's integral earlier. Now, I will take another example, minor extension of this problem. Instead of taking the wire to be rotating uniformly in the plane I will take it to be I mean having angular acceleration, so let us say it has an angular acceleration of alpha. So the rotation is not uniform with time, and will ask the same question as before and let us see.

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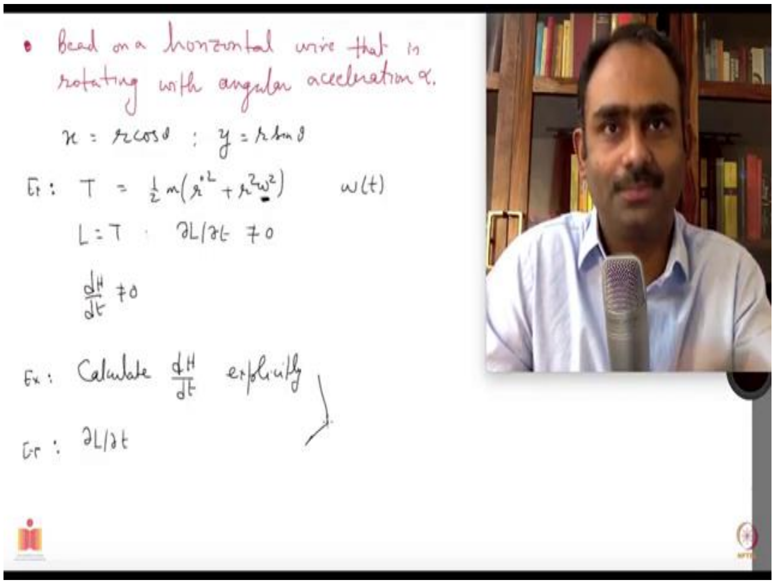

Let us do that color thing again, so example 2 is bead on a horizontal wire that is rotating with angular acceleration alpha, so that is the thing we want to talk about now. Again as before your x is r cos theta, your y is r sine theta you can find out theta given your initial condition and your alpha. So first you show what the kinetic energy is and your t will be again the same thing as before. So half m r dot square plus r square omega square.

But note now omega does depend on time, so you have a time dependence which is explicit because this omega is explicitly depending on time. See the r have time dependence but omega is not a generalized coordinate, it is not some theta or something, it is anyway one dimensional system. So this function because of this function your T has an explicit time dependence which means your Lagrangian which is same as t because again there is no potential energy.

Which means that del L over del t is not equal to 0 this time, last time it was. Now is H conserved? No, why? Because del L over del t is not 0, so your dh over dt is not equal to 0, so h is not conserved and anyway there is no chance of h being the total energy of the system, because there is an explicit time involved in the transformation from r to q which is good, so that is another simple example I will give you some exercise which you can do to get some practice with doing some minor algebra in this which is involved in this.

So what you do is calculate the total derivative dh over dt explicitly, so when you construct the h by the relation q dot de L over del q dot minus L, you take that 1 then you take a total derivative of it. So you have to use chain rule, do that, get whatever answer you get then you calculate del L over del t. So take the Lagrangian, take the partial derivative and show that dh over dt del over del t del L over del t they are same. So these two should give you the same answer.

So please do these 2-3 simple exercises that I have given and in the next video what will do is we will start talking about symmetry due to space translation. So here we were looking at the symmetry of time translation and we saw certain things are at conserved and we want to see the equivalent things when there is symmetry under translation in space that will be goal for our next video. See you then.