Introduction to Classical Mechanics
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Lecture 12
Conservation laws Conservation of Energy

In this course on several occasions we have talked about the symmetries of system. The system that we study and in this video and the next one and maybe next to next we would start talking about how symmetries that are present in the system provide us with certain conservations laws. That is the thing which we want to have in this video but before I talk start talking about symmetries and corresponding conservation laws, I want to settle few minor things. So, I will first talk about two things one is what is called Euler's theorem a very simple theorem and also about Einstein's Summation Convention, so that is how we will start this lecture.

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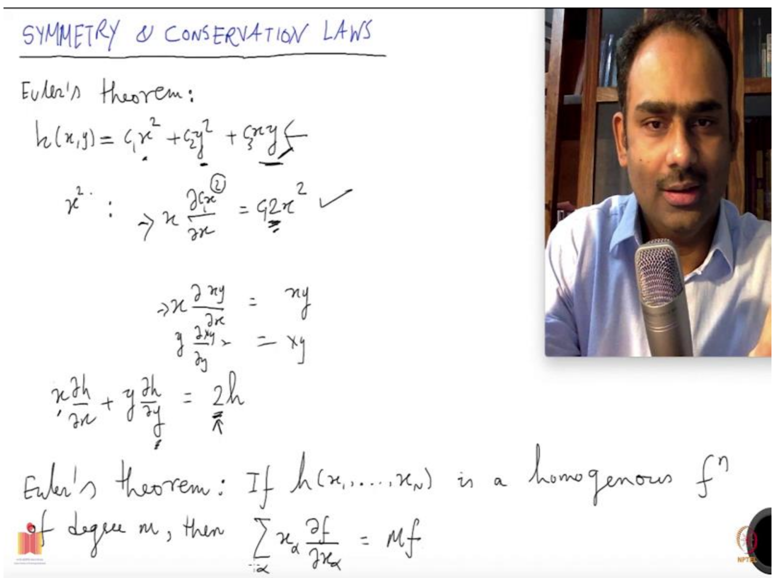

As I said I want to talk about Euler's theorem. So, idea is simple imagine you have some function which is a polynomial function. Let us say two variables to begin with. Let us say I have some function h of let me make h slightly simpler x, y. And I am looking in right now at a function of degree two, so h is polynomial in degree two so I could have something like this.

You could have x square y square plus x y you can multiply some coefficients here. Now, even though it is, not even though I mean it is a degree two polynomial but another special thing about this polynomial is it is homogeneous. It is a homogeneous function meaning each term is of degree two so this has two powers here. This also has two powers, this also has two powers because one power of x, one power of y, so this h is homogeneous function.

If I add a term like c3x here, then this h is no longer a homogeneous function. So, let us take a homogeneous function. Now, if I take a partial derivative with respect to x for example. So, let us take del h over del x. When I do so I remove one power of x from each term. So, from here one power x will be gone, form here it will be gone meaning this will give you 0. This one has one power of x so that will be gone and you will be left only with y.

So, that is what happen whenever you take a derivative with respect to one of the variables, that power is gone but if you multiply x you put that power back. So let me, let us say we take x square so what I am saying is del x square over del x is 2 x but if you multiply with x again you 2 x square which is what you had originally except for this factor of 2. And this factor 2 you have got because the 2 here. Let us look at what is happens when you look at this term.

So, again by taking a partial derivative I remove a factor of x but when I multiply the x again that is back and I get x y. So, you can already see probably that if I take a function which is homogeneous of second degree than the following is going to happen. if you take in this case for h x del h over del x plus y del h over del y then you are going to get 2h. Because when you do this you get a 2x square. I mean if there is a c1, factor c1 here your c1 will be again here.

And then when del x, del over del x on this it gives 0 but when you do a partial derivative with respect to y. It will generate another y square term in addition you will have a factor of 2 and this one here which you may realize that this will not create a power of 2 by taking partial derivative with x or with y but when you add them because your adding here this addition will create the 2 power. So, here you will have y del by del y of x y that will again generate x y.

And then when you add this two you get 2xy so it is clear that if I take partial derivatives and multiply with those relevant variables. And sum over all the variables I get the function back and what shows up here as a coefficient is the degree, degree of that polynomial or that function. So, clearly if this was a degree three function, a homogeneous function of degree three you would have gotten a three here. That is what is Euler's theorem.

So, it says, if h with let us say lots of several variables x n is a homogeneous function of degree n, degree m then if you take the partial derivative with respect to all these variables and multiply them with the same variable again and you sum over all variables then will get the same function back and the coefficient will be the degree of that function and that is Euler's theorem. That is one thing which I wanted to talk about just hold on for a second.

So, another thing is that you see I am writing down lots of summation symbols like here. So, I have to write these summation symbols lots of time in the appear very frequently in your equations so they make your equation look not so simple. So, what I will do is I will stop writing the summation symbols and I can that is interesting. So, I did something here let us see I should remove it perfect.

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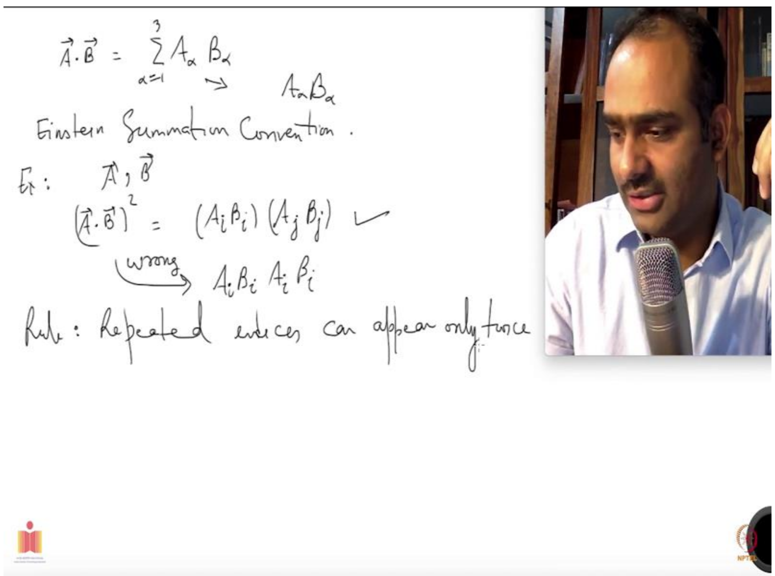

So, you see if I have to write let us say a dot product A Dot B, A Dot B two vectors A and B in three dimension let us say. Then I will write this as A alpha B alpha and summation alpha runs from 1 to 3. That is what I have been doing till now, things of this sort. But really I do not need to write this summation symbol because I can make a short hand notation and say whenever such a summation appears I will just drop it and write simply A Alpha B alpha. And I will remind myself that there is a summation over alpha because I can notice that there are the alpha appears twice. So, whenever an index appears twice I will know I was summing over it.

And this is a very powerful thing which is I mean for what I am saying it is not evident why it is so powerful right now it just look like a simple way of or neater way of writing things but this becomes very powerful when you are dealing with tensors. So, dropping this summation symbol is call Einstein’s, Einstein’s Summation Convention. Which just says if an expression you see a repeated index there is a summation implied there. If the summations have been dropped, maybe I will give you a simple exercise to do.

Let us say you have two vector A and B which are given to you and if I asked you to write A Dot B square using Einstein’s Summation Convention how would you write it. A dot B is Ai Bi. So, I have the index i getting repeated twice. Now, I have another A Dot B still remaining because there is a square I should write again Aj Bj. Now, you see I have not written Ai Bi again I have written Aj Bj and what you have to do is you have to convince yourselves that it would be disastrous to use again the index i.

So, meaning if instead of writing this you had written Ai Bi Ai Bi you get a wrong answer and it is obvious but I will leave it to you to figure it out. So, here we make a rule once you have done this exercise you will understand why I am making this rule. So, you make a rule that any index should not appear more than twice because if it does then you are making mistakes then your, whatever you are writing is not correct. So, that will be a rule.

Can appear at most twice, only twice so you will have if any index is repeated meaning if there is a summation over it then it should appear in that expression only two times not more. There may be some cases where an index is repeated but comes more than three times or more than two times but then there, we will make a specification, we will tell clearly that this index is repeated and there is a summation over this index also.

But if I do not say anything and if you see a repeated index twice then there is a summation, if I do not specify anything and you see a repeated index appearing more than two times then I have made a mistake. That will be what it is, so hence fourth I will never write a summation symbol. So, good so I will use Einstein’s Summation Convention and also use Euler's theorem for homogeneous functions and use them to look at our kinetic term which we had in the Lagrange, that is what we will do.

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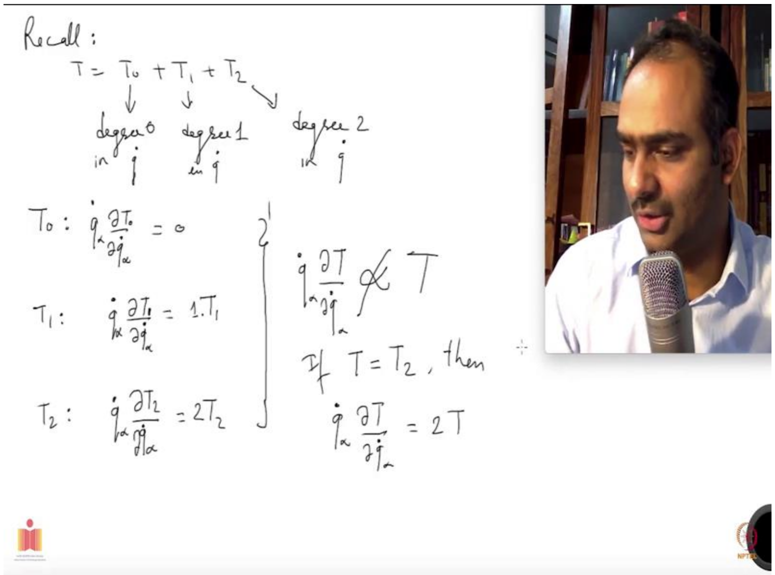

So, let us go to maybe next page recall that the kinetic energy term in the Lagrangian when you are using generalize coordinates has the following form the T is T0 plus T1 plus T2. Where T0 is of degree 0 in generalized velocities. I hope you remember this degree 0 in generalized velocities so q dots. This guy is linear meaning degree 1 in generalized velocities and T2 was is quadratic or degree 2 in q dots and maybe I can write that is fine.

So, let us look at this expression one by one so look at T0 term first, there is not much to see because maybe let us first look at T1. Let us look at first T1, so if I take T1 then if I take a derivative of T1 with respect to q alpha dot. I remove the power of alpha dot and I multiply q alpha dot here insert that power back again and if there is a summation implied because the alpha is repeated twice here. So, there is a summation here and you know sorry I am looking at T1.

You know that you will get T1 back, the coefficient here which is multiplying is 1 because is a function of degree 1. Let us look at this delta T2 which is quadratic alpha dot q alpha dot again there is a summation implied and you will know what you will get T2 and here it is there is no dependence on generalized velocities, so this one q alpha dot there is no need to do the summation it is 0 anyway.

That is what we get which means in general if you are taking a derivative of q alpha dot, this will not equal to, this will not proportional to T, this will not be proportional to T. It will not be proportional to T because this guy becomes two times T2 this guy become T1 and this guy is 0.

It cannot be proportional to T in general. There is one thing but if let us say for some reason which we will talk about soon your T was just T2. Let us say your T had only the quadratic pieces. So, if T is equals to T2 then q alpha dot delta T over del q alpha dot that will be two times T, if this is the situation, perfect.

Let us proceed, so now what I want to talk about is time translation symmetry and we will see what follows from there. So, imagine you have a system that is isolated from everything else which is not been influence by something else. So, imagine you have some set of particles which are interacting with each other in some manner and but there is no external influence on that system. So, there is no external agency which is exerting forces on this.

So, you know when that, the Lagrangian of this system would not depend explicitly on time. It would depend implicitly on time through the coordinates of the generalized coordinates or Cartesian coordinates for in whichever manner you are writing, the time dependence will be implicit through the coordinates but not explicit. Which is what you expect from isolated system and that is let us say imagine, let us say we have such a system.

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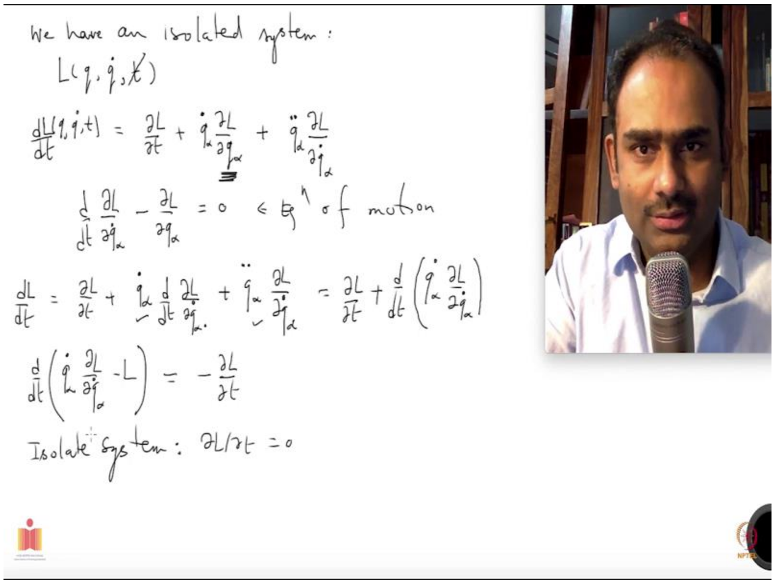

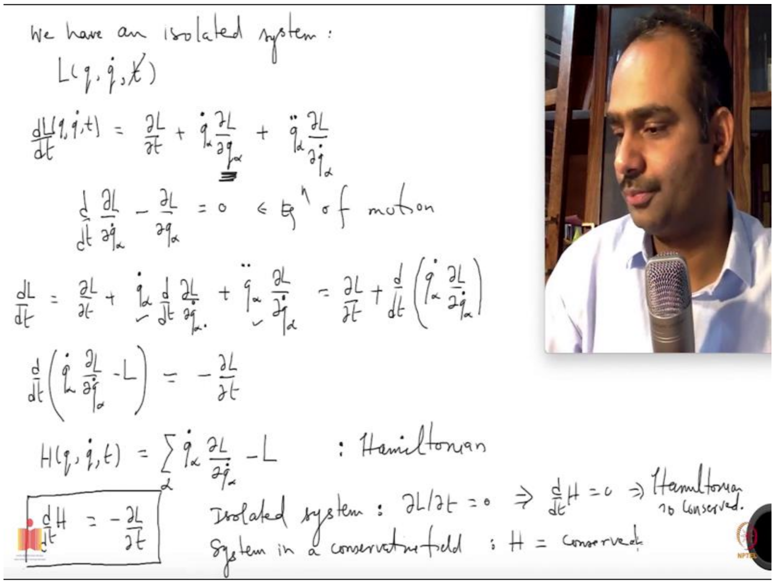

So, let us say we have an isolated system. And if you have such a system then your Lagrangian, so by q I mean all the entire set of generalized coordinates and q dot entire set of generalized velocities. I am not talking about any one-dimensional system this is a multi-dimensional it has several degrees of freedom. So, if this is a case then you will not have a dependence on T and let us see what the consequence of this is. To do so I want to look at the total derivative of the Lagrangian.

So, I want to look at what dL over dt is. So, for now what I will do is even though I am interested in such a system I will keep it general. So, I will keep L to depend on q, q dot and t and later I will put the time, I will remove the time dependence. For now, will keep general so and the system maybe something is influencing from outside all those things are here now. So, del L over del T so I just do the chain rule and get del L over del t plus del L over del q alpha q alpha dot.

Now, I should also differentiate L with respect to q alpha dots because L is the function all q and q dots. q alpha double dot and Einstein Summation Convention is being used right now. Now, this relation by itself cannot tell you anything. It cannot tell you because I have not put an information on how the system evolves with time. For that I need to put the equations of motions into this. So, I am going to now throw in the information about how system evolves meaning I will use Euler Lagrange equations here which says that del L over del q alpha dot d over dt minus del L over del q alpha is 0.

So, I am assuming right now that all the forces that you have are conservative, so I can write them in the Lagrangian itself and on the right-hand side you do not have any leftover generalized forces. So, I am imagining a system with conservative forces. So, that is the equation of motion which I want to substitute here meaning I will, what I will do is I will take this term del L over del q alpha and substitute this. So, this is the total derivative of L with, total derivative of del L over del q alpha dot so that is what I am going to put in here.

So, it says d L over dt equals del L over del t plus q alpha dot d over dt delta L over delta q alpha dot plus q alpha double dot q alpha dot. I hope that you can see these two terms, this one and this one, both these terms can be combined together. It is clear so I write del over del t of the Lagrangian, the partial derivatives then d over dt of q alpha dot del L over del q alpha dot. Is it clear? So, when this total derivative acts on q alpha dot it generates q double dot and this one remains, so this is a second term.

And when you take the first term, leave the first term without acting with the derivative and the derivative acts on the second term you get this piece, this one, so this relation is correct. And now you have total derivatives on both the sides of equation, so I can bring them together and write d over dt q alpha dot delta L over del q alpha dot minus L and that equals minus del L over del t and that is correct. So, I have taken the del L over del t on the other side.

Now, you see if your system is isolated then L does not depend explicitly on time meaning the partial derivative with respect to time would be 0, so the right hand side would be 0. This would be 0 for an isolated system. And if that is the case then this quantity would be conserved because the total derivative of that quantity in the round brackets will be 0, which means whichever way your system is moving, that quantity does not change its constant of integration.

So, it makes sense to define in general, not necessarily for an isolated system, let us say we define in general, this quantity to be H and this H depends on q whatever the generalized velocities are because they appear in the Lagrangian. So, L has q dependence, q dots L has t, so I define H q q dot t to be q alpha dot delta L over del q alpha dot minus L. And you are summing overall the, maybe this time I will just write.

So, this quantity is called the Hamiltonian of this system. Generally, Hamiltonian will have generalized momentum as its, in its argument but for now let us also call this as a Hamiltonian.

So, what you see is for a general system d over dt of H equals minus del L over del t, let me put a box around it. Now, I wanted to make the remark about isolated system. Now, if your system is isolated it is no time, explicit time dependence that is 0 which implies that H is a conserved quantity. So, d over dt of H will be 0 meaning the Hamiltonian is conserved, is conserved. That is good.

Not only that you do not, I mean even if your system was not isolated, even if it was interacting with another system, or let us say if your system was put in an external field which does not depend on time, now if it does not depend on time or it is correspondingly its potential does not depend on time, then the Lagrangian will not depend explicitly on time and del L over del t, the partial derivative of L with respect to time will still be 0 and again you will have Hamiltonian as a conserved quantity for that system.

So, even for a system in a conservative field, you still get H to be conserved. And note that your H is a first integral of motion, because remember what the functional form of first integrals was, your first integral depends only on generalized velocities and coordinates and time possibly. It does not depend on, it is not something involving a second derivative of time. So, which means if I give you an expression with H equals to constant then this is of differential equation of first order.

So, you immediately get a first integral of motion, if you know that this system has a symmetry of time translation. That is one nice thing but we still need to know or would like to know what H is physically. From here it is not evident what H is and that is what we want to do to figure out what H is and it is not, that is not hard. Let me see how I want to do it. That is good, so here we go.

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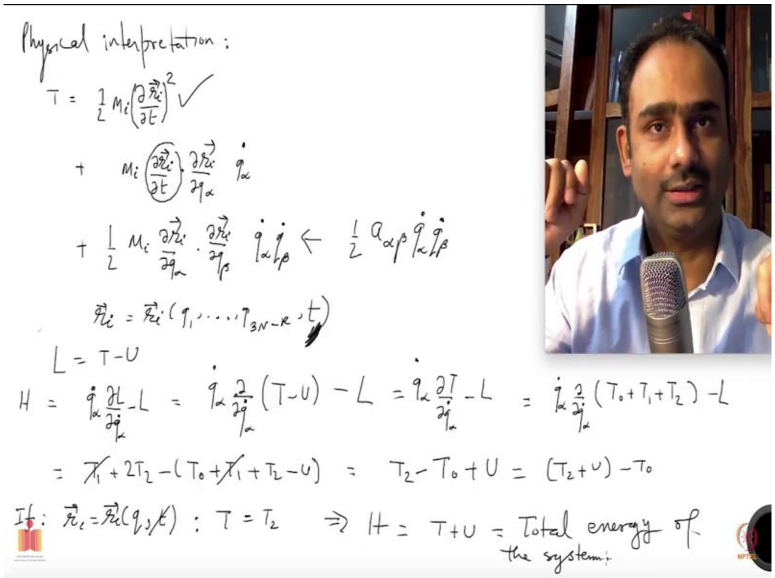

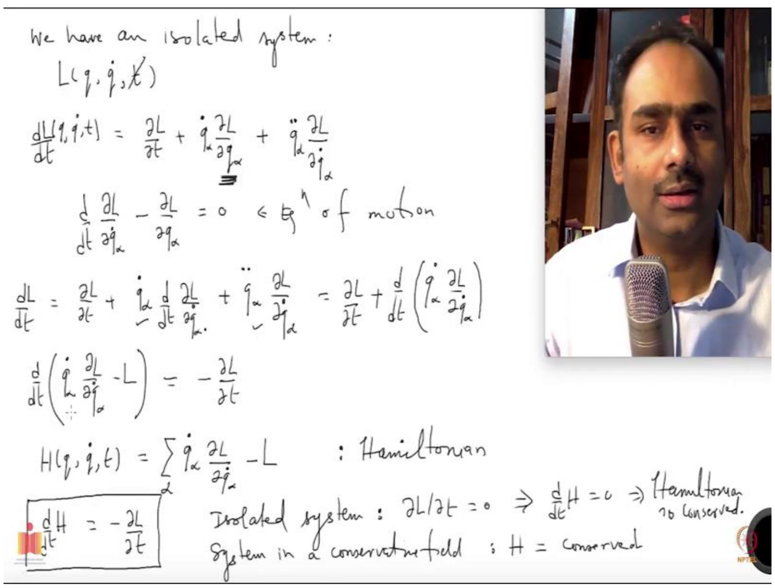

So, let me remind you what, physical interpretation of H. So, as we said that the T, the kinetic energy term is T0, and T0 let me write down, if you go back to your notes you will find this expression. delta ri over delta t and r is a function of the generalized coordinates and velocities, sorry generalized coordinates in time, Mi i is the label of the particle and you have a half. I omit putting a summation because the i, the i index i is repeated twice, so it is clear that there is a summation involved here, your kinetic energy is scalar, so this term has to be scalar and clearly this is being dotted into itself.

Then you have a linear term and the linear term was Mi delta ri over delta t dot delta ri over delta q alpha and you have a q alpha dot. So, I have two summations involved in here, the index i is repeated twice, actually thrice, so this is one on the cases where you will see that, even though it is appearing thrice there is nothing wrong. So, this is perfectly fine and if you have not seen such things before, make sure that you sit down and convince yourself that this is fine, there is no issue, it is not violating our rule which we announced that an index cannot appear more than twice.

And then your alpha is also repeated, so there is a sum over alpha also, that is good and then you have Mi is a half delta ri over delta q alpha delta ri over delta q beta q alpha dot q beta dot. And sometimes I will write this as, this piece, this piece as half a alpha beta times q alpha dot q beta dot. This is, I am writing only for the quadratic term, the last, last piece.

So, just to remind you your ri's, r r, too many times r is involved q1 to whatever the number of degrees of freedom your system has which we were writing I believe 3N minus k, yes. And in general there could be a time dependence as well. That is good. Also recall your L is T minus U, and now construct H. Now let us look at the Hamiltonian, your H is where is it, here q alpha dot minus L.

So, let me write it down, and let us see what it will be, q alpha dot del by del q alpha dot that is good. Now, your L is T minus U, so let me write it here for once minus L. Now, your U does not depend on generalized velocities, so I am not looking in those cases where your potential could depend on generalized velocities, so I am not looking at, for example electrodynamics. So, that is not being considered here right now.

If that is the case, then this is simply q alpha dot del T over del q alpha dot minus L. That is fine, Now, this becomes q alpha, now we will use our Euler's theorem, which says that if you take a partial derivative and insert q alpha, that coordinate again and sum over all the coordinates, you will get, if the function is homogeneous then you will get the same function back multiplied by the degree of that function.

So, when this acts on T0, let me write it, let me write it like this way, q alpha dot del over del q alpha dot just being, just writing everything explicitly here. So, what do we get, we get when this acts, it gets 0, so this term is gone, when this acts on T1 you get T1 back, because T1 is homogeneous degree 1, so you get T1. When this piece acts on T2, Euler's theorem says you get 2 T2 and anyway you have minus L, L is T minus U which is T0 plus T1 plus T2 minus U, the potential energy. Which becomes, so I have a T1 which cancels, there is a T2, a minus T2, so you get a T, sorry T2 and minus T0 plus U.

I hope that is correct. Let me see, perfect or let me write it as T2 plus U minus T0. So, nothing (())(38:51) eliminating when you have an arbitrary system but imagine your transition from Cartesian coordinates to generalized coordinates does not evolve, involve time, so let us say this time dependence is not there. If that time dependence is not there, then this term will be 0 because it involves partial derivatives, this term will be 0 because of the partial derivative involved here and this term will survive because there is no partial derivative, derivatives with respect to time.

So, in that case your T will be homogeneous degree 2. So, if let me see, yeah so if there is no, if there is no explicit time dependence, so let me remove the T, then your T is equal to T2, only the quadratic piece. If that is the case then your Hamiltonian will be T2 or which is same as T now plus U because T naught is gone and T2 is T, that you know, that is kinetic energy plus potential energy, which is the total energy of that system.

So, let me write down here, total energy of the system. So, in general H, the Hamiltonian does not have this interpretation of total energy but whenever your system does not have explicit time dependence in the Lagrangian and also your, when you are going from the Cartesian coordinates to the generalized coordinates, you will not require any explicit time dependence in going from r to q's. Then your Hamiltonian is the total energy, in that case you can interpret the Hamiltonian to be the total energy.

I think I will stop this video here and next time we will take simple example where I will do some explicit calculations, so about the Hamiltonian, so that we get some practice and also doing the calculations and also get an understanding of what the Hamiltonian is. So, that is what will be the plan for next video and we stop this one here. See you in, see you later.