Introduction to Classical Mechanics
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Cyclic Coordinates
Lecture No. 11

Hello let us start today's lecture by summary of what we have done till now and we will after the summary will take up some new things today.

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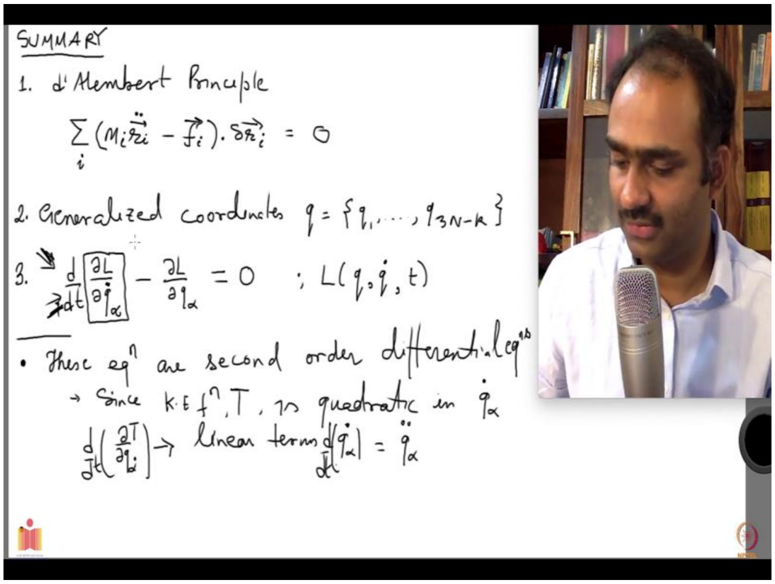

So here let us summarize, summary. So, we have started this lectures by writing down first the d'Alembert principle and that was mi ri double dot minus the forces, forces other than the constant force times the virtual displacements that are consistent with the constraints. That is what d'Alembert principle is. And then we went on to introduce generalize coordinates, the reason was to transform this part into generalize coordinates, independent generalize coordinates, so I could equate the coefficient of d q alphas to 0.

So, I introduced generalized coordinates q of q's, so the set is q 3N minus k. Then we derived Euler Lagrange equations, which are the equations of motion, the described the evolution of system and they were the following del L over del q alpha dot d over dt, that is a total time derivative minus del L over del q alpha, let us say all the forces are obtainable from a scalar potential so that the forces are consecutive. Again then I can write this otherwise on the right hand side you have generalize forces as well. Our Lagrangian is a function of generalized coordinates, generalize velocities, the term derivatives of q are called generalize velocities and t also.

Let us make some observations now, so up to here what we have done till now and we took some examples. So point number 1, note that this Euler Lagrangian equations, these equations are second order differential equations. So, these equations I am referring to Euler Lagrangian are second order, second order differential equations, eq, second order differential equations while the second order let us look at here the Lagrangian we wrote down last time you remember we wrote down last time the most general form of kinetic energy function the t and we saw that it has a T0, T1 and T2, T2 is quadratic in the generalize velocities.

So Euler Lagrangian t minus u will have quadratic terms then when you take a partial derivative here this this piece with respect to one of the generalized velocities, you take the quadratic function and you take a partial derivative with one of the velocities will be left with a function which will be linear in the generalize velocity. And then when you take the time derivative you will generate the second order differential of q, the second order time derivative of q and that is why these equations are second order differential equations.

Let me, maybe let me write me down here, since the kinetics energy function T that is what we have been calling it, is quadratic in q alpha dots you have you remember you have terms like q alpha dot q beta dot time of function which is a function of the q's. This is what we have discussed I think in the last video. Because of this and taking a time derivative, taking a partial derivative with respect to q alpha dot of T which is the part of Lagrangian will will generate a linear term in alpha dot, linear terms in, terms which are linear in q alpha dot.

And then when you take the d over dt which is here you will be taking d over dt of this this guy which is nothing but q alpha double dot. You see that your equations of motion are second order differential equations which you already know that they should be, also note a very important property here that in writing down these equations of motion, the Euler Lagrangian motions these ones, we never appealed to any special property of the generalize coordinates. There was nothing special about the coordinates which gives us this form.

If instead of choosing q alpha, we had chosen some other set q alpha prime we would have still gotten the same form of the Lagrangian equations, Euler Lagrangian equations we would still have gotten this which means that the form of Euler Lagrangian equations is independent of the choice of the generalize coordinates that you make. Which is quite clear form our procedure we never appeal to any special property of the q and more precise, let me write it down, maybe write down on next sheet.

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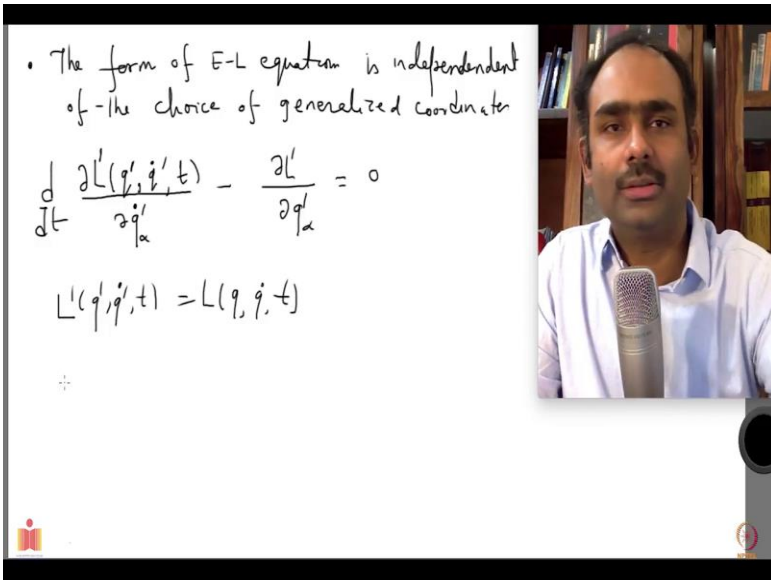

The form of Euler Lagrange equations is independent of the choice of generalized coordinates that is quite evident statement. Never the less let me write down what it means, mathematically it means that if you had chosen instead of the q's, q primes I am making a short hand notation instead of writing q1, q2, so forth, I am just writing q. If we had chosen this, then still you are going to get the following, del over del q alpha prime dot d over dt minus del L over del q alpha prime equals 0 that is what you would have gotten.

Where you know your Euler Lagrangian is the kinetic energy minus potential energy, take L in in terms of r and t the Cartesian ones convert them to the generalize coordinates q or the q prime so all these three L's are the same and that is what you see here, so let me put L prime here. So, what I am saying is your L prime q prime q prime dot t is same as L q, q dot t, these are the same functions. So, that is what I mean by saying that the equations the form of equations is independent of the choice of the generalize coordinates, that is one important thing. Let us see what I should say, maybe to the next video to the next slide.

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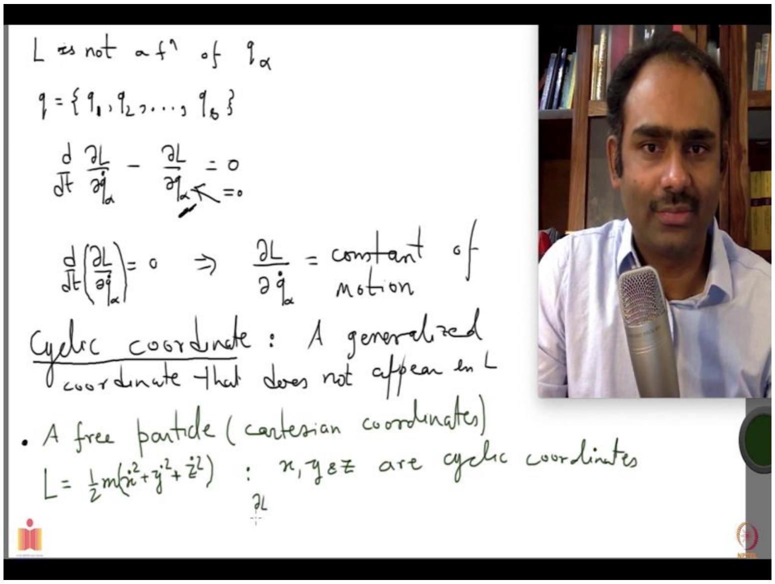

Now let say you are studying some system and you have setup the Lagrangian for that system and you discovered that not all coordinates appear in the Lagrangian. Those coordinates which do not appear in the Lagrangian are called cyclic coordinates or ignorable coordinates and there is something nice you can deduce if such a case arises and that is what we want to see now. So let us lets write it down, so let us say you are given a Lagrangian L and your generalized coordinates are q1, q2 s, whatever that s is, s is the degree freedom you have, that is what your degrees of freedom are.

Now let us say the Lagrangian is not a function of, let me put this over here. Let us say your L is not a function of one of these coordinates, let us call that coordinate q of alpha. So, that alpha could be 1 or 2 or whatever, so let us say one of them does not appear in L. If that is the case, then let us see what we can say from the equations of motion. So equation of motions says, d over dt del L over del q alpha dot so I am looking at the equation of motion corresponding to that particular alpha right now, the one which is not appearing in the Lagrangian minus del L over del q alpha equals 0.

If q alpha does not appear in the Lagrangian then this partial derivative is 0. So this guy, this term is 0 which means d over dt of del L over del q alpha dot is 0, it is nice. It is nice because it says that if you look at the quantity del L over del q alpha dot if you look at this guy then that guy remains constant, you see I am looking at a total time derivative here meaning as a time changes in system evolves things are moving around no matter what is happening, this guy del L over del q alpha dot that remains a constant.

So let me write it down, it says del L over del q alpha dot is a constant, it is a constant of motion. Your system is evolving things are moving around but the value this has does not change with time. So, let me write down clearly what all I have said I have defined already a cyclic coordinate. What is the cyclic coordinate, is a is a generalized coordinate coordinate that does not appear appear in the Lagrangian. When I say coordinate does not appear, it does not mean that the velocities cannot appear, the velocities can appear.

The the statement about cyclic coordinate is only about the coordinate not their respective velocities. So, that is what I mean by generalize coordinate and we see that the moment there is a coordinate which is cyclic, you have a corresponding constant of motion. So, this guy is conserved del L over del q alpha dot, that look like a little abstract right now but we can reveal what this guy is if we look at our simplest example of, let us say of free particle, that is what we always do.

So, whenever I want to test something I take the simplest cases or if I want to understand something which is new to me, I take that new thing in the context of the simplest cases I already know. So, I will take this and look for a a single particle in, which is moving freely so there are no forces and let us see what is the equivalent of this guy. So, I take the example of I would like to use green color if I am taking an example and I think I can do it, looks good.

So I take a free particle and I use Cartesian coordinates, I can use Cartesian coordinates are also a set of generalized coordinates, in this case Cartesian coordinates. I am assuming that particle is accessing all the three dimensions. So, what is the Lagrangian maybe I should go to next, fine no problem. So what is the Lagrangian? The Lagrangian in this case is as you already know very well half m x dot square plus y dot square plus z dot square and you see that the coordinates x, y and z they do not appear in Lagrangian the derivatives appear the velocities appear but not the not the coordinates.

So in this case, your x, y and z are cyclic, they are cyclic coordinates or they are ignorable coordinates, which means that there has to be a constant of motion which will be del L over del q dot where q will be x, y and z all of them, all three of them. So, let us look at what del L over del x dot is, so if I look at del L, let me go to next, how do I go, yes, fine.

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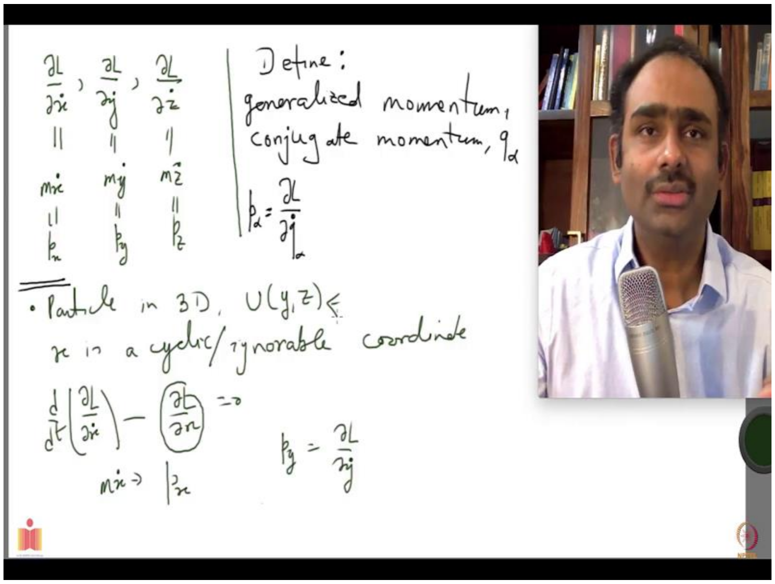

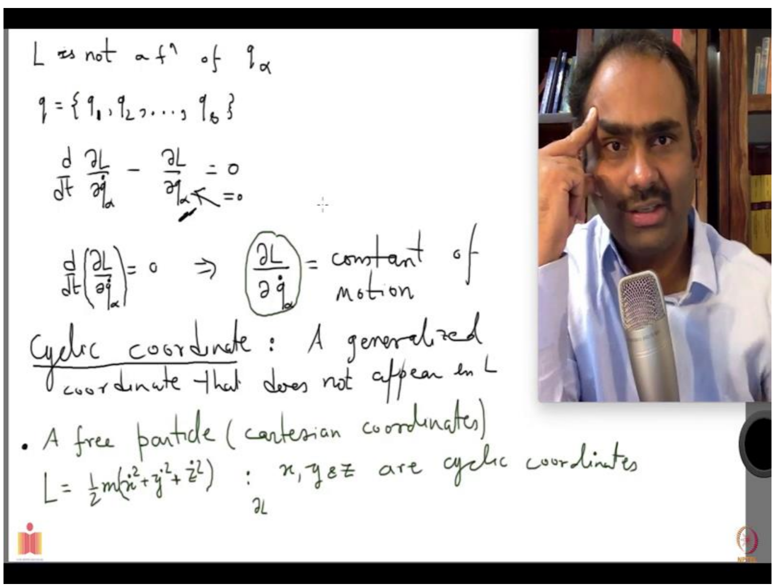

So let me look at, I want to do this, I want to calculate this thing for our case. So I should be looking at del L over del x dot and also del L over del y dot because that is also a cyclic, y is also cyclic coordinate and del L over del z as well because z is also a cyclic coordinate and what are these, your del L over del x dot is mx dot, your, you have x dot square so you take derivative you get x dot mx dot this will be my dot and this will be mz dot.

And what is this, this is mass times x component of the velocity, so that is the momentum in the x direction, momentum in the y direction and that is the momentum in the z direction. So, you see that in this case of a particle moving in three dimensions, what is constant is the momentum, all the 3 component of the momenta are constants which you already know that the momentum is going to be a conserved for this particle where there are no forces. So, this leads us to define a new quantity what we call as generalized momenta.

So, I will define let me make it black again, so I define generalized momentum or also we call it canonical momentum or conjugate momentum as well, let me write conjugate, conjugate momentum. So, I am defining a generalized momentum or conjugate momentum corresponding to q alpha to be here del L over del q alpha dot. This q alpha could be any of the any of the coordinates it does not have to be a cyclic coordinate.

So, corresponding to every q alpha I define del L over del q alpha dot to be the generalized momentum and I denote it by p f alpha. And in our example for a free particle here we have seen that the corresponding conjugate momenta are constants they do not change, change with time. Let me take another example here. Now let us take again a particle and for simplicity I will take it to be moving in two dimensions and it is let us say also experiencing a force on it.

Particle in 2D I take, so x y plain let us say and on which you have a force which is given by a scalar potential and this time I take the potential to depend only on y and z, I am saying it is moving in x y direction, so let us say it is in fact y, do this. So, let us say we take a particle in 3D and we take the scalar potential to depend only on y and z.

Now if this is the case then your Lagrangian does not depend on x, because it is only through the potential at the, your x will appear, which means that for this case x is a cyclic coordinate or ignorable coordinate. If that is so then something has to be a constant of motion, something has to be conserved and that is the corresponding momentum, conjugate momentum.

So, let us look at what the corresponding conjugate momentum is and it is clear what is it, it is just you have look at del L over del x dot. And why it is going to be conserved, for this reason which we have already seen del L over del x, del L over del x is 0 because this no x here. So, this guy is conserved and we have already seen what del L over del x dot is, it is mx dot.

So, px the component of p in the x direction that is conserved but from here you also see that if I took, if I look at p y which is del L over del y dot then that guy is not 0, that guy is not a constant because you will see that del L over del y from here will get a contribution because of the u and you already know that because there is a force which will act on the particle in y and z directions, remember the force is a gradient of potential the momentum in y and z directions will not be constant but in this case there is no force in the x direction.

The gradient of u, the the potential is not changing in x direction so there is no force in the x direction and that is why the component of momentum in x direction is a constant. This is what you also expected already in this example. Let us see a slightly non trivial example, only slightly, till now I have been just using Cartesian coordinates, so let us see an example where I am using generalized coordinates and the simplest one to think of is again that of a free particle let us say confine to plain, x y plain and I use polar coordinates and then again we see what we can say here.

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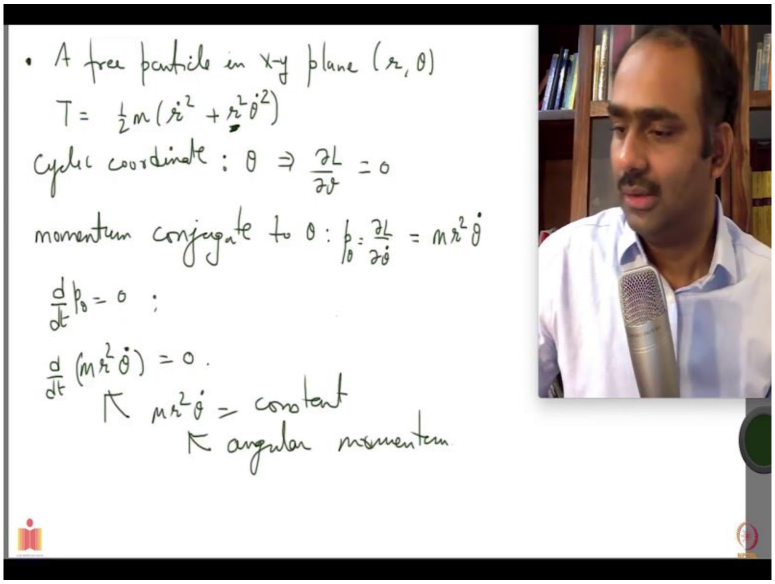

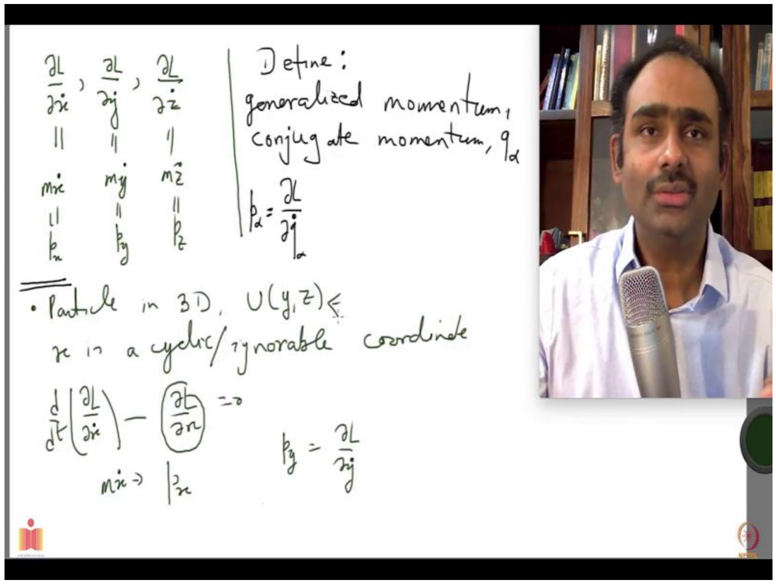

Another simple example, and because it is a example it has to be green. So, again I take a free particle in let us say x y plane and I used the polar coordinates r and theta. We already seen it that the kinetic energy is half m r dot square plus r square theta dot square. So, which coordinate is a cyclic coordinate in this case? Clearly, theta is not appearing here, so theta is a cyclic coordinate. Theta dot is there but that is the velocity that is not a coordinate.

So, theta is a cyclic coordinate which means del L over del theta is 0 and you see your r is not a not a cyclic coordinate, r appears here so del L over del r is not 0.

So, let us look at the conjugate momentum, momentum conjugate to theta and that is del L over del theta dot let us go back del L over del q alpha dot, so your q alpha is theta now so del L over del theta dot. And what is that del L over del theta dot so this term is gone, gives 0. This term will give you half m times r square and theta dot square when you take a derivative gives you 2 theta dot, so you get m r square theta dot and this is constant because of your Euler Lagrange equations d over dt let me call it p theta.

And your p theta is 0 or d over dt of m r square theta dot is equals to 0 we have got another, not another we have got a constant of motion in this case and I hope you have already recognize this quantity. What is m r square theta dot? m r square theta dot, you recognize this, m r square theta dot is the angular momentum.

The angular momentum of a particle which is not acted upon by any any torque will be constant and that is what you see here so this is nice we, in this case the the quantities are very familiar but even if let us say we are looking at a system which, where which we have very crazy kinds of generalized coordinates. All you have to do is first always, first start by searching for the coordinates which are cyclic, you just see which one does not appear in the Lagrangian.

Once you have found it you already know that whatever is happening to the system whichever way things are moving around, there has to be a corresponding conjugate momentum that will be conserved and that is a nice thing to know. Let us let usmake a summary of what we have said in I will make some observations based on that.

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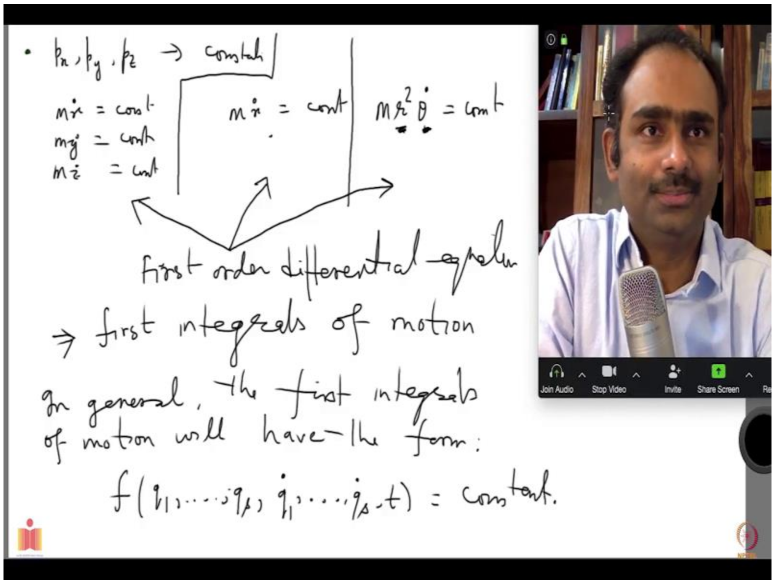

So in the 3 examples I have taken, I have seen that the example number 1 where we have, I had a free particle moving in 3 dimensions, I had px, py and pz, they were all constants. And px let me write, I wanted to write it this way mx dot is equal to constant, my dot is equal to constant, mz dot is equal to constant, that is what we saw in the first example. And then I had a particle which was in a potential which depend on x and I saw that mx dot was a constant in that case.

And then we look at the example of again a particle in two dimensions but using polar coordinates and we saw that mr square theta dot was a constant. Now note all these equations, all these equations that I have written down here, mx dot constant, mr sqaure theta dot constant, they are all first order differential equations, there is only one time derivative involved, theta dot or x dot or y dot or z dot.

And these equations I have not obtained, these first order differential equations I have not obtained by solving the system first, it is not that I have solved for the system found where, how things are evolving with time where, how q alpha, where alpha runs from one whatever degrees of freedom you have, I have found the time dependence of q alpha and then found by taking a derivative of this q alpha these equations, that is not what I have done.

What I have done is, looked at which coordinates are cyclic and immediately written down these first order differential equations. And usually these are the things which are much more informative because they immediately tell you something about the system, see you are not always very much interested in knowing where each particle is going. That will be of course useful and you have the full information but that may not necessarily be very interesting thing to know.

What maybe more interesting thing to know is for example that the angular momentum is conserved in this case or the momentum is conserved or some such quantities are conserved and those you can immediately get. And these first order differential equations, let me write down, these are first order differential equations. And these first order differential equations are called first integrals of motion that is what they are called first integrals of motion.

And in general if you are looking at system of s degrees of freedom, you would have the first integrals of the following form, the first integrals of motion will have the form, of course it can involve only first order time derivatives, so it will be some function f like here it depends on the coordinate r and first time derivative of theta.

So, it would be in general function of the coordinates let us say the system has s degrees of freedom and then it will involve the first derivatives, first order derivatives and possibly time and this will be a constant. That is where we will stop today and I hope you realize that having first integrals of motion is a very useful thing you immediately get some good piece of understanding about your system, we will continue from here in the next video, see you then.