Introduction to Classical Mechanics
Professor. Anurag Tripathi
Assistant Professor
Indian Institute of Technology, Hyderabad
Lecture 10
Kinetic term in generalized coordinates

So as I was saying, we will start looking at the kinetic term, when we are using generalized coordinates, we will see what are the differences compare to using Cartesian coordinates.

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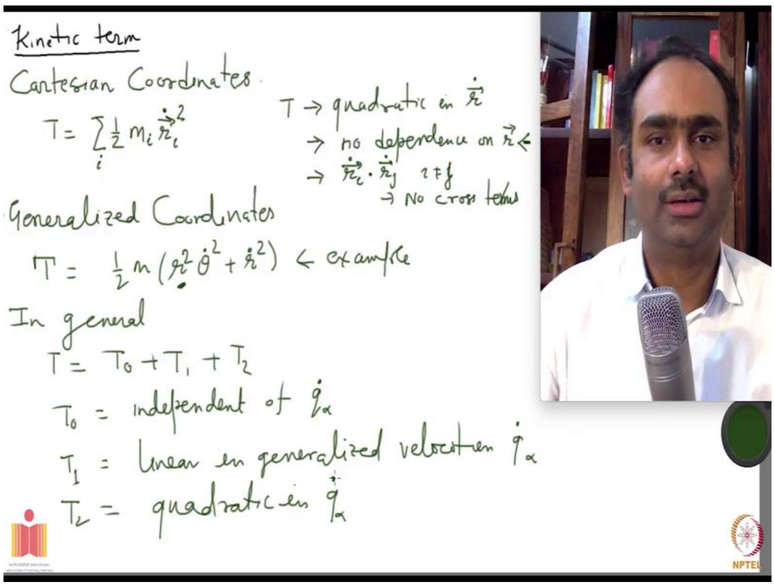

So, first what do we see when we see, when we work with Cartesian coordinates. Actually it will be nice to have color, let see if I can do it quickly, I can, good, good, I think it is done, Cartesian. So, what do you see when you are using Cartesian coordinates, you get T as half mi, let me put r dot, half mi ri dot square, that is what it is. So, this is of course quadratic in velocities, note that there is no term which involves the product of velocity of particle i with a particle j, there is no such term. So they are all, T is really diagonal in the velocities.

Also note that there is no factor in here, which depends on the coordinates itself. So, T does not depend on the coordinate but only on the velocities, this is another thing and also note that there is no linear term, it is all just quadratic. So, when you are using generalized coordinates, T is quadratic in r dots, note that there is no dependence on r and also for example there could have been r square dependence which is absent and also there is no, what I wanted to say, there is no ri dot rj dot kind of a term here, where i is not equal to j. You have only the terms where i and j are equal, so there are no cross terms, t e r m, no cross terms are present.

Let see what the situation is when we start using generalized coordinates. Let see if I can make, I do not know what I am making it colorful but let see. Okay good. Now I do not know how to get the green back, that is sad. Yeah here is the green, let us look at the generalized coordinates now. Yeah we have already looked at T in at least few contexts and you remember when we were looking at particle which was moving in two dimensions XY plane and we were using polar coordinates.

At that time, we wrote the kinetic energy of that particle in polar coordinates and it was half m r square theta dot square plus r dot square. So, in this example itself you see that this condition is, this not condition, this thing is not true. Generalized coordinate r can appear or in principle, in some different context, some other coordinates could also appear. So, it not just the generalized velocities, theta dot and r dot appear, but also coordinate itself can appear which is the case here.

Here it is still of the form where cross terms are not present, so you still do not have a term like theta dot r dot. But as I will show that this is also not a general statement, in principle they can be present, that is one thing and also here you do not have a term which is just involving the velocity of one of the coordinates and as you are going to see very soon that is also not a general statement. So let us see, let see what the general expression would be, this was one specific example which we covered in one of the previous videos.

So, in general the kinetic energy can be written as the sum of three terms, which I will call as T0 plus T1 plus T2, where T0 is independent of q dots, independent of the generalized velocities. So let me put alpha and alpha will run from 1 to whatever degrees of freedom you have. So, it will be independent of all the generalized velocities and that is what 0 signifies. T1 is the term which will be linear in generalized velocities, linear in generalized velocities, velocities, which is q alpha dots. Then T2 will be quadratic in q dots. That is what we will prove now.

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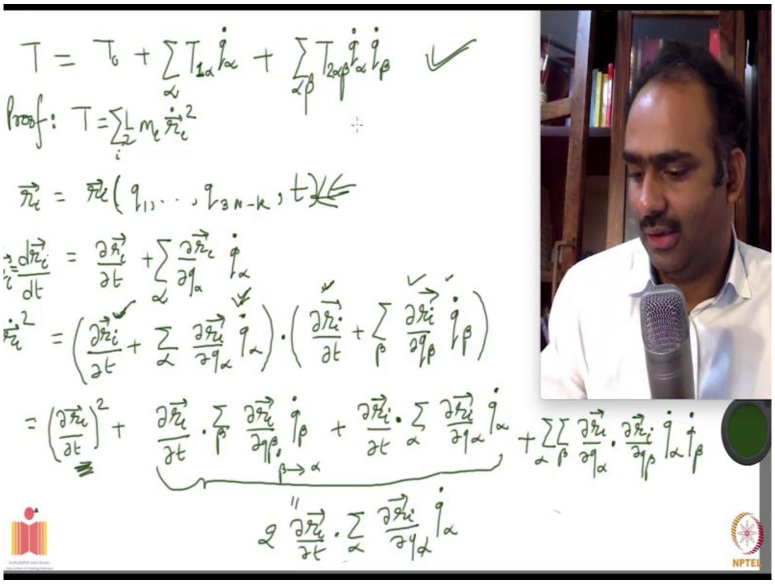

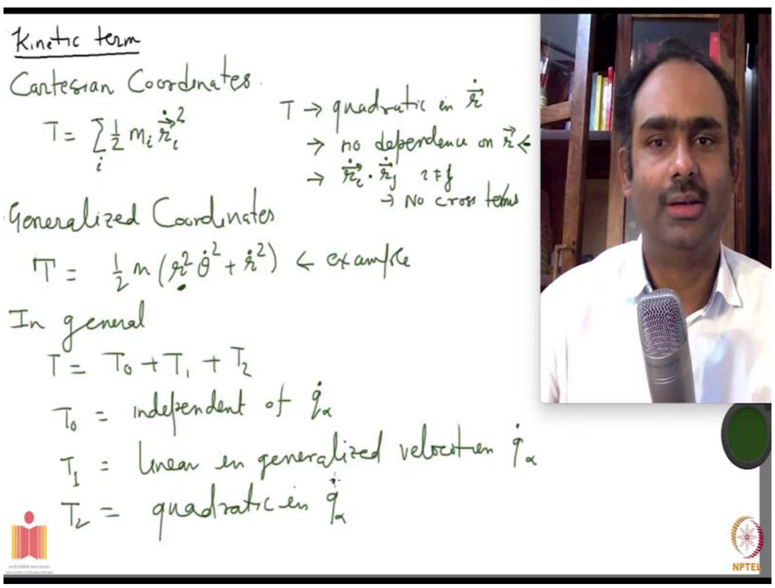

So, what I want to prove is this that, let us see, yeah, so I will introduce a notation and make what I said just now more explicit, that it will, T will have a form T0 plus I said that this will be T1 would be linear in velocities, so T q, T1 would be having a form like q alpha dot times some coefficients or some functions, let me call it T1 alpha, I distinguish this T1 alpha from T1 because T1 has no index and this guy has. So, it is clearly it is not the same quantity. So, you have T1 alpha and there will be a sum over all the alpha. That is good.

Then as I said there will be terms which are quadratic, so you could have q alpha dot, q beta dot and T2 alpha beta summation over all alpha and beta. So, whatever I said here is more explicitly written here and note that I am saying that it is not q alpha dot q alpha dot, I am saying q alpha dot q beta dot because cross terms may be present and this is what we want to prove now and proof is simple.

Proof, so what do I have to do, just remember that half mi ri dot square sum over all the particles, good. Now your ri will be related to your generalized coordinates by the following, ri q1 to whatever q you have, let us say 3N minus k, total number of degrees of freedom and it could also possibly depend on time. Your constraints of, constraints on the system may be changing with time, so your transformation from r to the generalized coordinates would involve time as well. That is good.

All I have to do is take this, take the time derivative, so that I get an r dot and take the r dot square and put in here, that is all I have to do which is not difficult. So, dr just a second, hold on please, so the velocity is dri over dt which is del ri over del t, so you take the partial derivative and then you take the partial derivative with respect to all the coordinates, alpha ri and you have q alpha dot and you should sum over all the alphas. That is good.

Now I take this which is ri dot and square it, ri dot and square it. So, I write this as del ri over del t plus the same thing which you have above, then I should dot it with again the same piece delta ri, remember i is the level for the particle, delta T plus summation delta ri over delta q alpha q alpha dot summation over alpha.

Now if you have not already noticed the mistake and if you have not noticed still despite might (())(12:47) that there is a mistake, you need to know one thing that this is going to give a trouble if I use the same index alpha which is here and the same index alpha here. I should be very careful and I should not use the same index on both the sides.

So it will be wiser to change the alpha to beta, q beta, q dot beta, beta and I leave it to you to figure out what will be the problem if I use alpha in this piece and alpha in here also, you should know why you will get into trouble. That is good, now I should take the dot products or have been, yeah. Now this, so I take a dot of this with that which is easy, r over delta t square. Then I will take a dot product of this piece with this term, del i, this piece with that term and you will have another similar term coming from this one and this one and they are both same because the alpha is dummy.

I am just summing over all possible alpha. So in these two products, this times that and this times this, I can use the same indices, maybe I will go slowly so that you can see plus let us multiply this with that one. So, I get delta ri over delta t, that is correct dot summation over beta delta ri over delta q beta q beta dot and then you look at the product of this guy with this guy which is delta ri over delta t coming from here dot summation over alpha delta ri delta q alpha q alpha dot.

This dot product is because of the two vectors here, that is good and then you have one more term plus product of these two, these two will give you term which is quadratic in the velocities. These ones are generating terms which are linear in velocities, there is the only one velocity involved here, one velocity involved here plus so I write down summation over alpha summation over beta delta ri over delta q alpha dot because there is a dot product here, delta ri delta q beta and then you have q alpha dot q beta dot, q alpha dot q beta dot.

Now what I was saying earlier was that this beta here can be replaced by alpha, it does not matter, you are just summing over all the coordinates and multiplying with the relative, corresponding velocities. So, I have replaced this beta by alpha, then these two become identical and the sum will have a factor of two which is, so these two together become 2 delta ri over delta t dot summation over alpha delta ri over delta q alpha q alpha dot. So, this is what I was claiming here, first term no velocities are involved in here but this is a function of q and t, right now I am writing r as a function of q and t.

So, this is a function of q and t but there are no velocities involved in here. So, that is belongs to T naught. Then this piece has only linear dependence on the generalize velocities. So, that is what is here and so you can read off from here after summing over all the masses and all the particles, your T1 alpha and this is the one which is going to give contribution to T2 alpha. So let us see, I will write this down what happened. So, here we have or maybe I can let see if I can do it there itself. Let us, yeah maybe not, let us be neat and clean.

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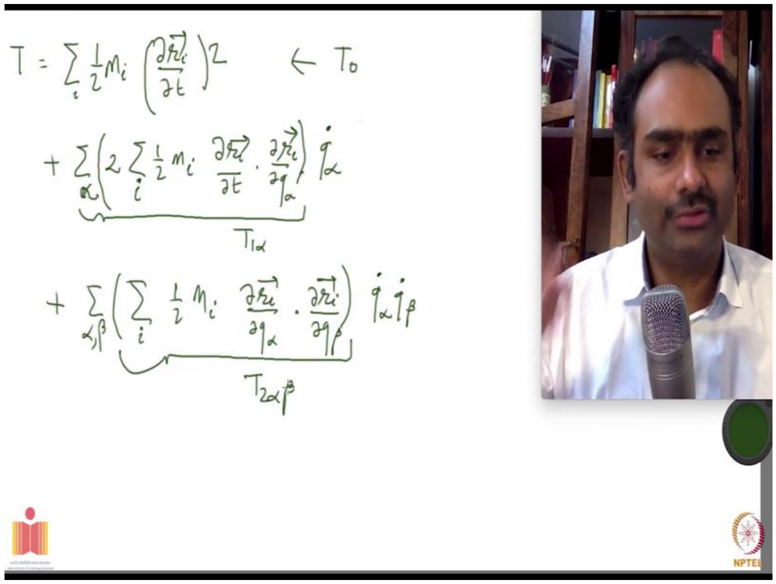

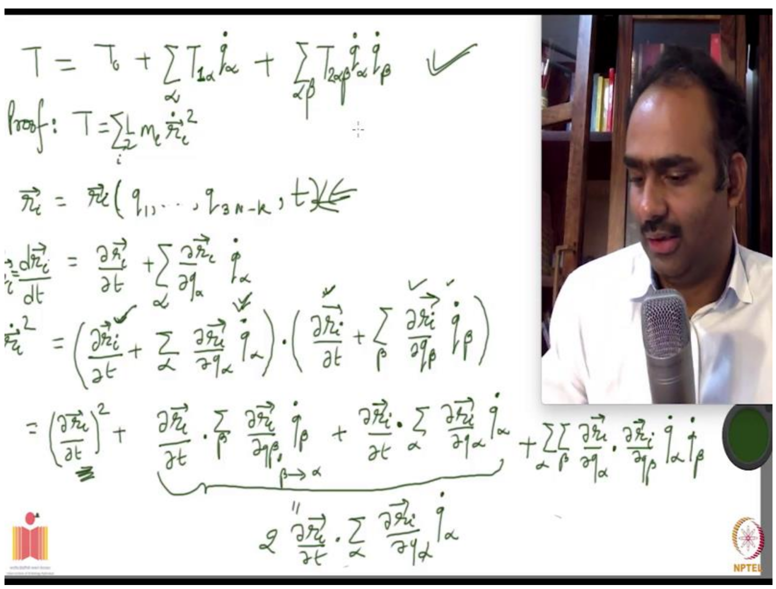

So I will write T as, I am just now bringing in the half mi which was remaining and I have to sum over all the particles. So, if you look at what we had earlier, I had dt square. Let see, ri del over del t of this square, so that is the first term which is what your T0 is now. Let us look at the second term now, good, I will multiply this with half mi and sum over all the is. So, you will get summation over i 2 times half of mi delta ri over delta t dot delta ri over delta q alpha and this is the summation over alpha which I can take out and I can put the summation over i in here and put the summation over alpha here, this entire thing with q alpha dot.

So let us check, q alpha dot, this summation I have pulled out and the sigma over i, I have pulled in. So, this is the piece which is, this piece, this entire thing is your T1 alpha, good plus the term which is quadratic in velocities and here you will have summation over alpha and beta then summation over i half mi and we had delta ri over delta q alpha q beta, very good, q alpha dot q beta dot and this is what your, this piece is T2 alpha beta, let us see what did we call T2 alpha beta, yes.

So, I have proofed what I was claiming that you have all kinds of terms here and note that these coefficients T1 alpha and T2 alpha beta and T naught, they depend on generalized coordinates and time in general and there may be cases depending on what kind of coordinates you are using that some of the coordinates may not appear and some of velocities, generalized velocities may, will generalized velocities may not appear linearly but those will be specific to those cases. But that is the general result in front of you. We will stop this video here and we will meet next time.