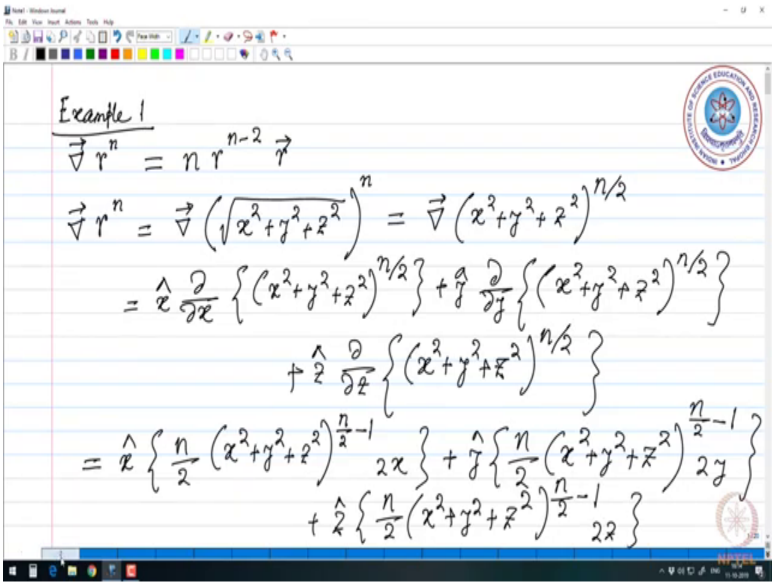
Electromagnetism
Dr. Nirmal Ganguli
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Lecture - 08
More problems on vector differential calculus

Hello, earlier we have learned about the differential calculus in vectors. We have learned the operations like gradient divergence and curl and we have also learned second order derivatives. Now, we will work out some examples involving gradient divergence curl and second order derivatives. Those of you who are already familiar with these things and have worked out a lot of examples have solved a lot of problems, you can skip this part. It would be useful for others, I would nevertheless recommend everybody to go through this part as well.

(Refer Slide Time: 01:19)



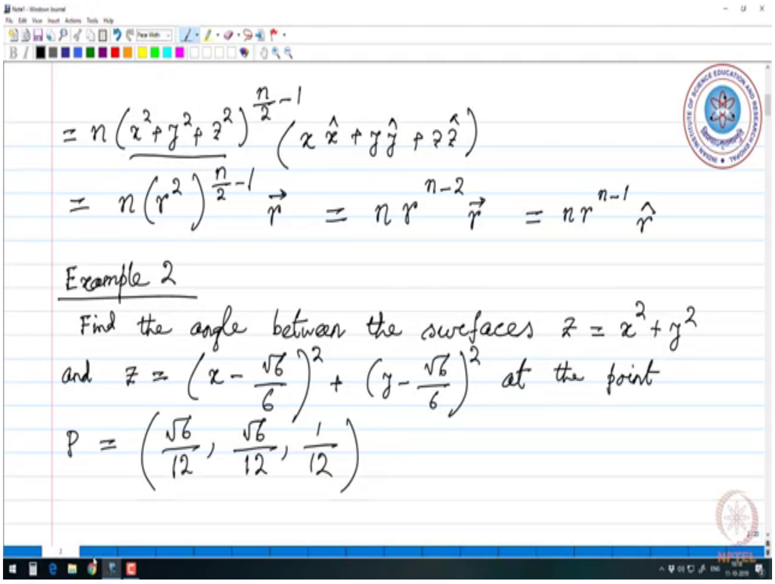
So, let us start let us consider few problems; few examples and let us work that out. In the first example, we consider a gradient. We take the del operator operating on the position vector magnitude power n and if we do that, we are supposed to show that this quantity is equals n times r power n minus 2 times r vector. This is the gradient that is something we are supposed to prove. Let us work it out. The gradient of r power n can be written as the gradient of x square plus y square plus z square; square root.

This is the magnitude of the r vector position vector and this one power n which is nothing, but the gradient of x square plus y square plus z square power n over 2. This with the gradient operator can be expressed as x cap del del x of x square plus y square plus z square power n over 2 plus y cap del del y of the same thing plus z cap del del z of the same thing.

Now, if we work this out; if we work out the derivatives, we will find that this quantity is equal to x cap n over 2 x square plus y square plus z square power n over 2 minus 1 times twice of x plus y cap times.

Similarly, n over 2 x square plus y square plus z square power n over 2 minus 1 multiplied by 2 y plus z cap n over 2 x square plus y square plus z square n over 2 minus 1 multiplied by twice z. This is the quantity.

(Refer Slide Time: 04:57)

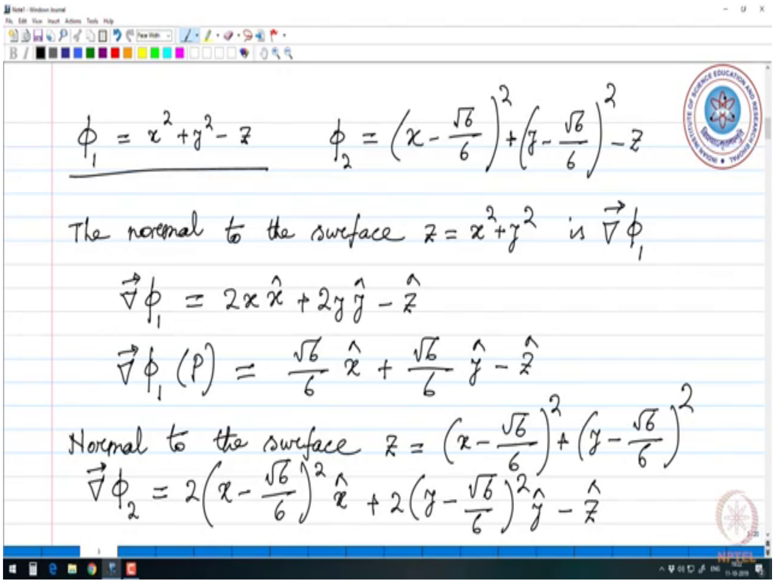


If we now simplify this, we will arrive at this quantity is equal to n times x square plus y square plus z square power n by 2 minus 1 times x x cap plus y y cap plus z z cap. And this is nothing, but n this quantity is position vector square r square power n by 2 minus 1times r vector which is n times r power n minus 2 r vector. It is essential essentially n r power n minus 1 r cap the r cap being the unit vector along the direction of r vector, the position vector.

Let us see the second example. In the second example, we have the equation of 2 surfaces and we are supposed to find the angle between these two surfaces. So, the statement of the problem is find the angle between the surfaces given by z equals x square plus y square and z equals x minus square root 6 over 6 square plus y minus square root 6 over 6 square and the angle is to be found at a given point; point P with the coordinates square root 6 over 12 square root 6 over 12, 1 over 12. This is the coordinates of the point.

So, how do we find the angle between two surfaces? The direction of the surface is given by the normal to the surface. And how do we find the normal to this these surface?

(Refer Slide Time: 08:31)

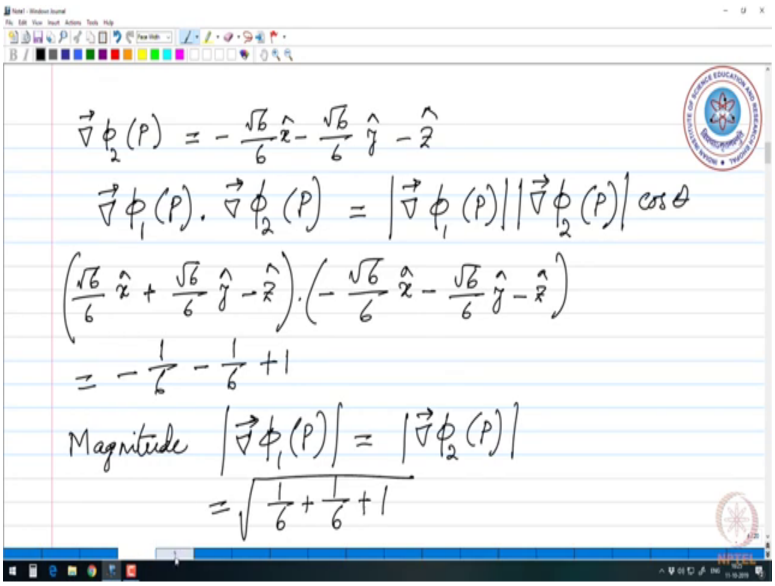


Let us see if we consider two functions phi 1and phi 2; phi 1 is given as x square plus y square minus z and phi 2 is given as x minus square root 6 over 6 square plus y minus square root 6 over 6 square minus z. Then if we have these functions, then we can say that the normal to the surface given by z equals x square plus y square the normal to the means the normal to the first surface can be given by the gradient of phi 1.

And if we work out the gradient for this function phi one we will find that gradient of phi one equals 2 x x cap plus 2 y y cap minus z cap and this is pretty obvious there. Then if this is the gradient, then the gradient of phi 1 at point p with the coordinate of the point p given earlier is nothing, but square root of 6 over 6 x cap plus square root of 6 over 6 y cap minus z cap.

Similarly, and we will have to find a normal to the second surface. The surface second surface is given by z equals x minus square root 6 over 6 square plus y minus square root 6 over 6 square. This one would be given by gradient of phi 2 and that is if we work it out, we will find twice x minus square root of 6 over 6 square x cap plus twice y minus square root 6 over 6 square y cap minus z cap. This is the gradient.

(Refer Slide Time: 12:15)



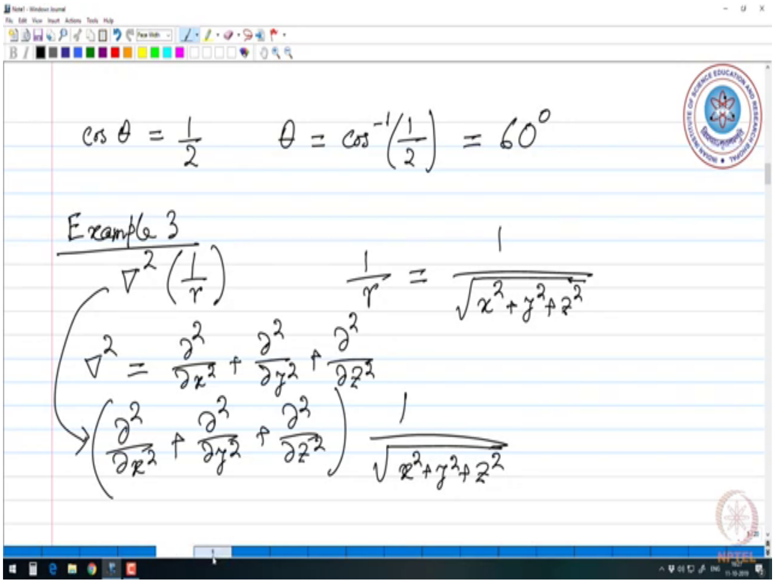
And if we now try to find the gradient at point P that is gradient of phi 2 at point O that is given as minus minus square root of 6 over 6 minus x cap minus square root of 6 over 6 y cap minus z cap.

Now, we are supposed to find out the angle between these two vectors. So, what are we supposed to do? Let us take a dot product of the gradient of phi one at the point P with the gradient of phi 2 at the point P and this quantity is equal to the magnitude of the gradient of phi 1 at point P multiplied with the magnitude of the gradient of phi 2 at point P times the cosine of the angle between these two vectors.

So, the dot product; let us calculate the dot product in its component form first, square root of 6 over 6 x cap plus square root of 6 over 6 y cap minus z cap. This vector dotted with minus square root of 6 over 6 x cap minus square root of 6 over 6 y cap minus z cap.

So, this gives us minus from the first term, we get 6 over 6 square that is 1 over 6 from the second term also gives the same minus 1 over 6 and the third term is plus 1. So, this is the value of the dot product we get and if we now work out the magnitude of the vectors, then the magnitude of the vectors will be nothing but, both vectors have the same magnitude and that would be square root of each the sum of each component squared. So, the magnitude of phi 1 at P equals to the magnitude of the gradient of phi 2 at the point P and that is 1 over 6 plus 1 over 6 plus 1 square root of this.

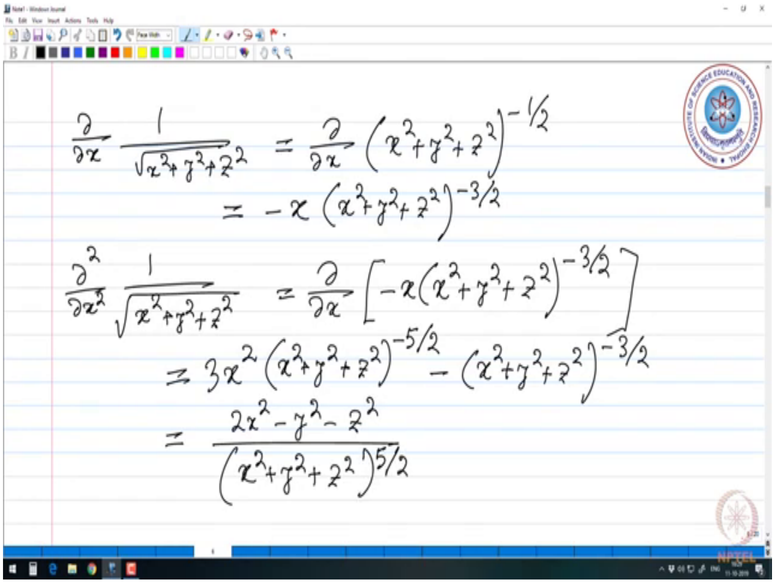
(Refer Slide Time: 15:49)



So, we can write down that from after working this out we will find that cosine of the angle between these two vectors that is cos theta is half and if cos theta is half, then theta is cos inverse of half and that is 60 degree. So, 60 degree is the angle between these two vectors. Let us move on to another example that is example 3. Example 3 involves evaluating the Laplacian of 1 over the position vector.

So, let us try evaluating this. The position vector 1 over the position vector is given as 1 over x square plus y square plus z square square root of this and we know the del of del square operator del square is del 2 del x 2 plus del 2 del y 2 plus del 2 del z 2. So, this quantity would be del x 2 plus del y 2 plus del 2 del z 2 of 1 over square root of x square plus y square plus z square. Now let us calculate each part of it separately.

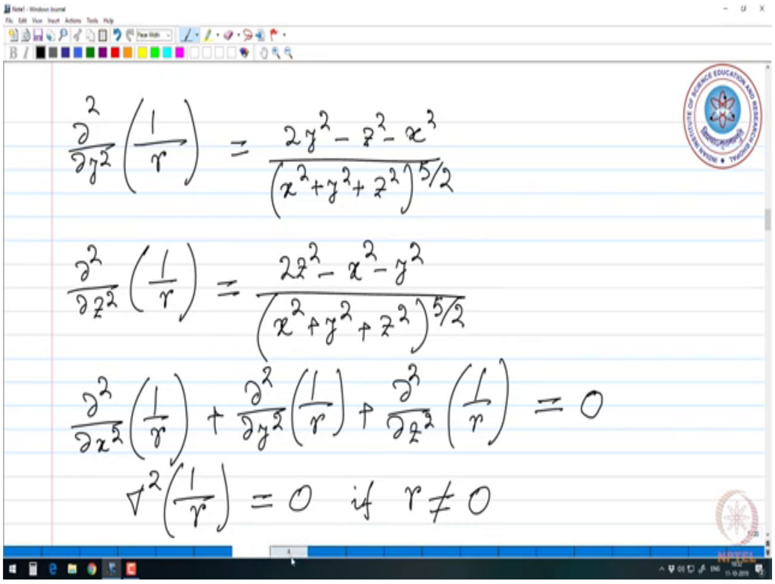
(Refer Slide Time: 17:47)



If we perform the first partial derivative, del del x of 1 over square root of x square plus y square plus z square, what do we get? We get this is; this is nothing, but the power of x square plus y square plus x square becomes minus half square plus y square plus z square minus half. This helps us evaluating it and if we now evaluate we will find that this quantity is minus x times x square plus y square plus z square power minus 3 by 2.

If we now, perform the partial derivative once more that is del 2 del x 2 of 1 over x square plus y square plus z square square root of this, we will get del del x of minus x x square plus y square plus z square power minus 3 by 2. And this turns out to be 3 x square x square plus y square plus z square power minus 5 by 2 minus x square plus y square plus z square power minus 3 by 2. And this can be written in a compact form as 2 x square minus y square minus z square divided by x square plus y square plus z square whole power 5 by 2.

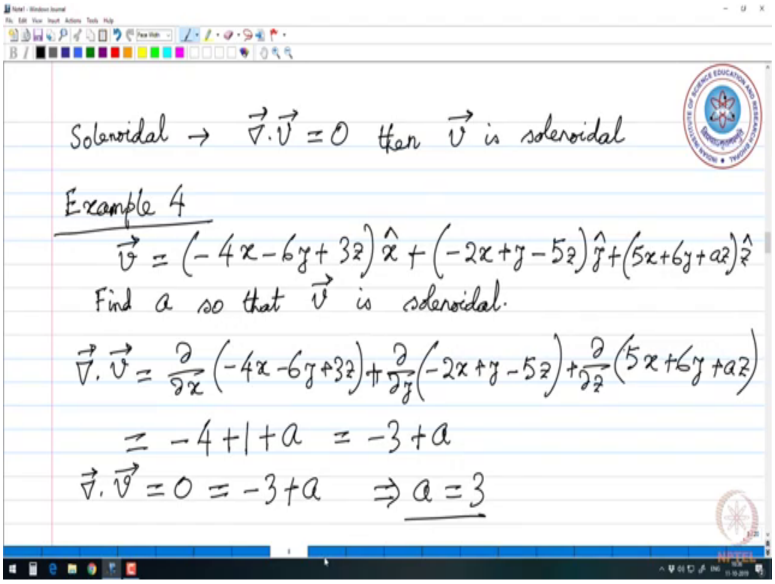
(Refer Slide Time: 20:07)



Now, with similarity we can write that del 2 del y 2 of 1 over r. This would be 2 y square minus z square minus x square over x square plus y square plus z square power 5 by 2 and del 2 del z 2 of 1 over r is similarly equal to 2 z square minus x square minus y square over x square plus y square plus z square power 5 by 2.

So, in order to get the result, we have to add all these terms; that means, del 2 del x 2 1 over r plus del 2 del y 2, 1 over r plus del 2 del z 2, 1 over r. And looking at the expressions for this, we see that this will become 0 so; that means, the Laplacian of 1 over r is 0, but there is a note of caution. This is 0 only when r is not equal to 0 if r goes to 0, then we will have to be more careful. This may be something else and we will see that kind of an example somewhere else.

(Refer Slide Time: 22:31)



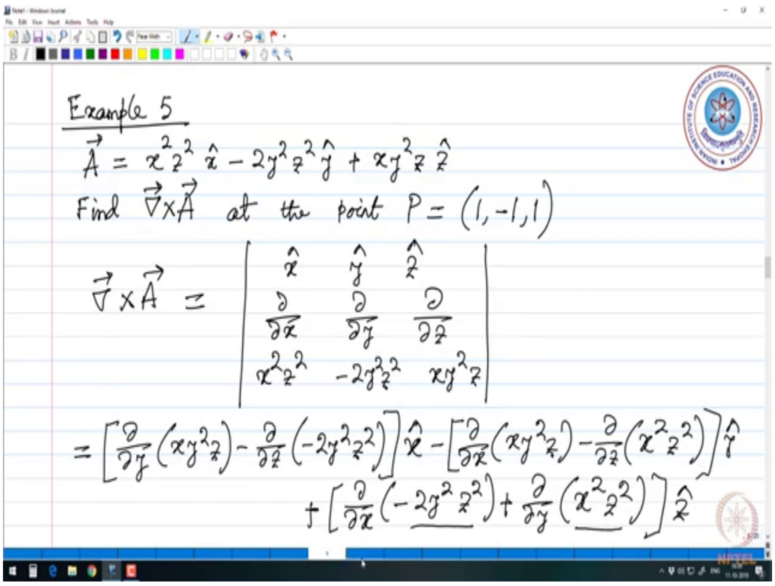
Now, let us consider another example. Before considering another example, let us get introduced to a term solenoidal. Solenoidal means the divergence; if the divergence of a vector becomes 0, then the vector is called solenoidal. And here in this example that is example 4, we are suppose we are given a vector v minus 4 x minus 6 y plus 3 z x cap plus minus 2 x plus y minus 5 z y cap plus 5 x plus 6 y plus a z z cap.

Now, the question is now we are asked to find a so, that the vector is solenoidal. Obviously, the approach would be to equate the divergence of this vector to 0 and in order to do that we will have to find the divergence of the vector first. Divergence of v is given as del del x of the x component of the vector that is minus 4 x minus 6 y plus 3 z plus del del y of the y component minus 2 x plus y minus 5 z plus del del z of the z component 5 x plus 6 y plus az.

And that becomes so, from the first term we get minus 4 and the rest becomes 0 because those are not functions of x. From the second term, we get plus one because the coefficient of y is 1 and the rest does not matter those go to 0 and from the last term, we get plus a that is minus 3 plus a.

So, in order for the vector to be solenoidal, we need divergence of v equals 0; that means, minus 3 plus a equals 0. This implies that a equals 3, this is the condition for the vector to be solenoidal.

(Refer Slide Time: 26:23)

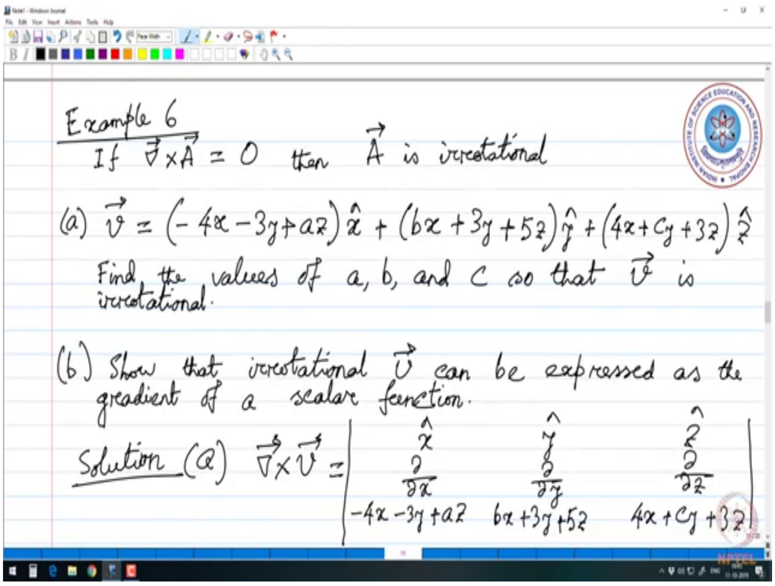


Let us move on to another example, example 5. Here we are supposed to find the curl at a given point. So, we have been given a vector A that is expressed in the component form as x square z square x cap minus 2 y square z square y cap plus x y square z z cap. We are asked to find the curl of this vector at the point, the coordinates of the point is given 1 minus 1, 1.

So, we know how to express curl in its determinant form that is x cap y cap z cap del del x del del y del de,l z the x component that is x square z square. Here is the y component minus 2 y square z square. Here is the z component x y square z if we evaluate this determinant, we will get the curl and we can evaluate it to find del del y of x y square z minus del del z of minus 2 y square z square x cap minus del del x x y square z minus del del z x squared z squared y cap plus del del x minus 2 y square z square plus del del y x square z squared and z cap is the direction.

Now, in the last term, we can see that this is not a function of x and the partial derivative with respect to x will go to 0 and this is not a function of y. So, the partial derivative of this quantity with respect to y will 0 go to 0. We will have no z component for this curve.

(Refer Slide Time: 30:07)



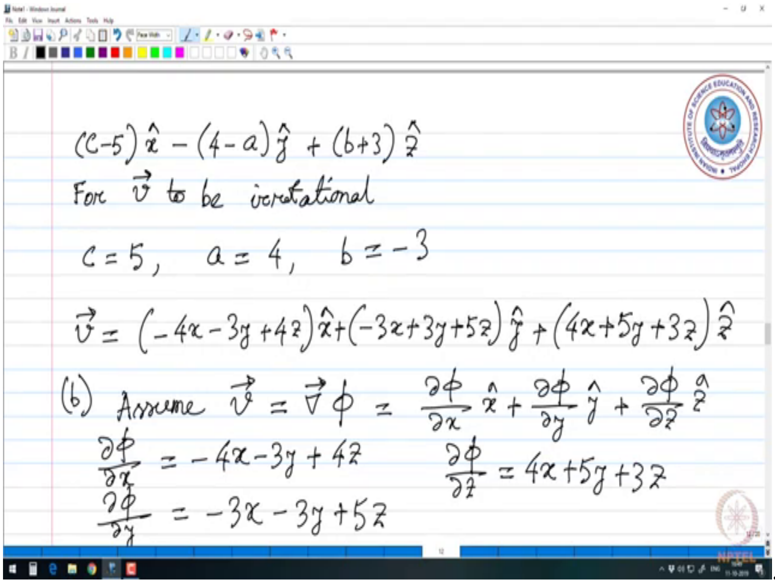
So, with this we can write that the curl becomes 2 x y z plus 4 y z squared x cap minus y squared z minus 2 x squared z y cap. Now, at the given point P, we are supposed to evaluate it.

So, the curl of a at point P equals 1 minus 1 sorry 1 minus 1 1; this will become 2 x cap plus y cap. This would be the value of the curl at the given point. Let us consider another example, this was example 5. So, we will try and have example 6 here a vector is called irrotational if the curl of that vector goes to 0; that means, if curl of a equals 0, then we call a irrotational.

Now, there are two parts of this problem in part a we are given a vector v as minus 4 x minus 3 y plus a z x cap plus b x plus 3 y plus five z y cap plus 4 x plus c y plus 3 z z cap. Now, we have to find the constants a b and c. So, that this vector is irrotational and in the second part of the problem we are supposed to show that when v the vector is irrotational. It can be expressed as the gradient of a scalar function. Earlier we have learnt that the curl of the gradient of a scalar field is always 0.

Now, here we are supposed to prove that if the vector v as expressed here is irrotational, it can be expressed as the gradient of a scalar function. So, in order to solve this problem, let us attempt the part a. In part a, the first thing is to evaluate the curl of this vector v and this would be given in the determinant form x cap y cap z cap del del x, del del y, del del z minus 4 x minus 3 y plus a z b x plus 3 y plus 5 z 4 x plus c y plus 3 z.

(Refer Slide Time: 36:13)



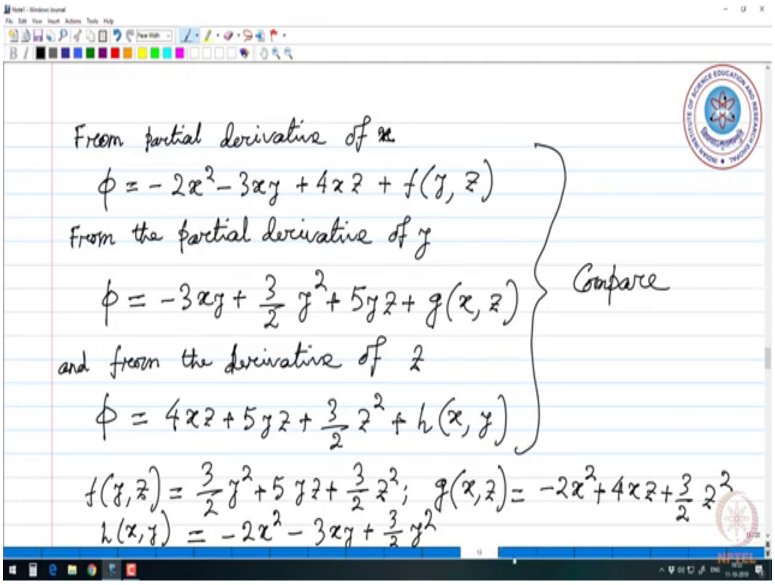
.

If we work out this determinant, we will get after carefully working this out; we will find c minus five x cap minus 4 minus a y cap plus b plus 3 to z cap. Now, for the curl to be 0 each component of the curl has to be 0 and that will give us for the vector v to be irrotational c must be equal to 5 a must be equal to 4 and v must be equal to minus 3 that will only satisfy the condition; that means, the vector v when irrotational would be given as minus 4 x minus 3 y plus 4 4 z x cap plus minus 3 x plus 3 y plus 5 z y cap plus 4 x plus 5 y plus 3 z z cap.

Now, let us solve part b; in part b, we assume first that the vector v is given as the gradient of a scalar function phi which means del phi del x x cap plus del phi del y y cap plus del phi del z z cap is the vector v and that will require each component of this gradient matches each component of vector v, then only this will be satisfied. So, from the form of vector v written above, we can write del phi del x equals minus 4 x minus 3 y plus 4 z del phi del y can be written as minus 3 x minus 3 y plus 5 z and del phi del z is given us 4 x plus 5 y plus 3 z.

And now, if we partially integrate them term by term; that means, in the first case when its partial derivative of x, we partially integrate phi adding a function of y and z as the integration constant with respect to x.

(Refer Slide Time: 40:23)



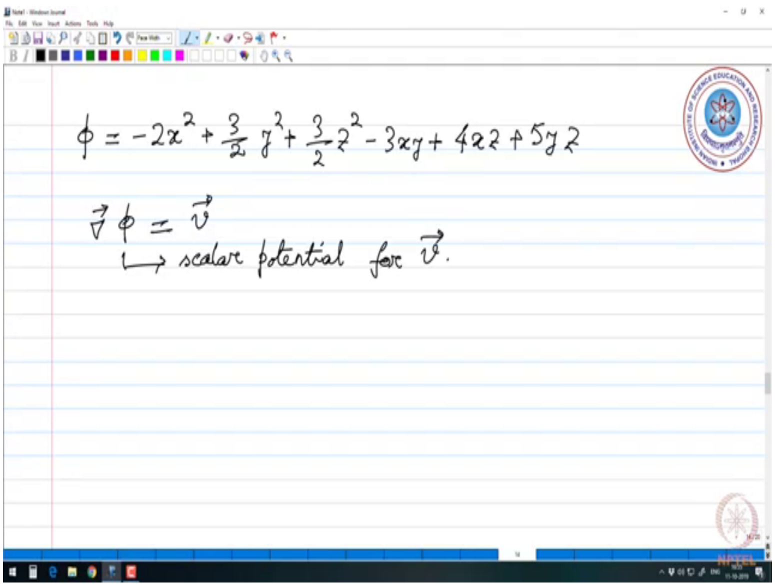
That means from the partial derivative of x, we can get we get phi equals minus 2 x square minus 3 x y plus 4 x z plus the constant of integration in x that may in general be a function of y and z.

Similarly, from the partial derivative of y, we obtained by partial integral phi is minus 3 xy plus 3 y 2 y squared plus 5 y z plus an integration constant with respect to y that is in general a function of x and z. And from the partial derivative of z integrating partially integrating that part, we will obtain phi equals 4 x z plus 5 y z plus 3 y 2 z squared plus an integration constant which is in general a function of x and y.

Now, we need to compare these three equations and get a form for these integration constants that are in general function of the variable that is variables that is that are not there in the partial derivative. And by comparing we can guess that the form of f y z may be given as 3 over 2 y squared plus 5 y z plus 3 over 2 z squared.

Similarly, g x z may be given as minus 2 x square plus 4 x z plus 3 y 2 z squared and h x y. This can be expressed as minus 2 x squared minus 3 x y plus 3 by 2 y squared. Once we have this we can again compare which are the terms we have counted multiple times and then correct for that.

(Refer Slide Time: 44:07)



After correcting for everything, we will obtain the expression for phi equals minus 2 x squared plus 3 by 2 y squared plus 3 by 2 z squared minus 3 x y plus 4 x z plus 5 y z. So, if we have this quantity phi as the as a scalar and if we take the gradient of this scalar, we will get the vector v for which the curl will be 0.

So, we have obtained the scalar quantity from which we can find the irrotational vector; that means, we can express the gradient of phi equals the vector v which means phi can be written as the scalar potential for v. That means, if we have any irrotational vector field, it can be represented by a scalar potential.