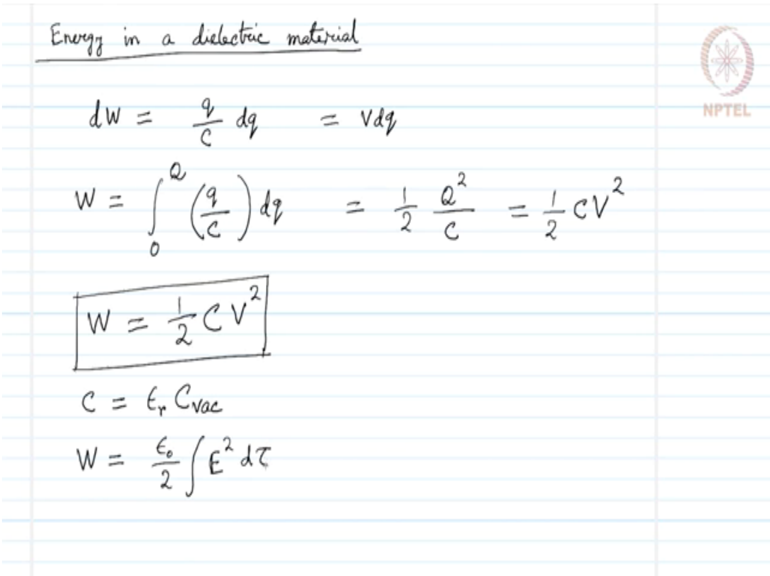
Electromagnetism
Dr. Nirmal Ganguli
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Lecture - 53
Energy in dielectric materials

(Refer Slide Time: 00:38)



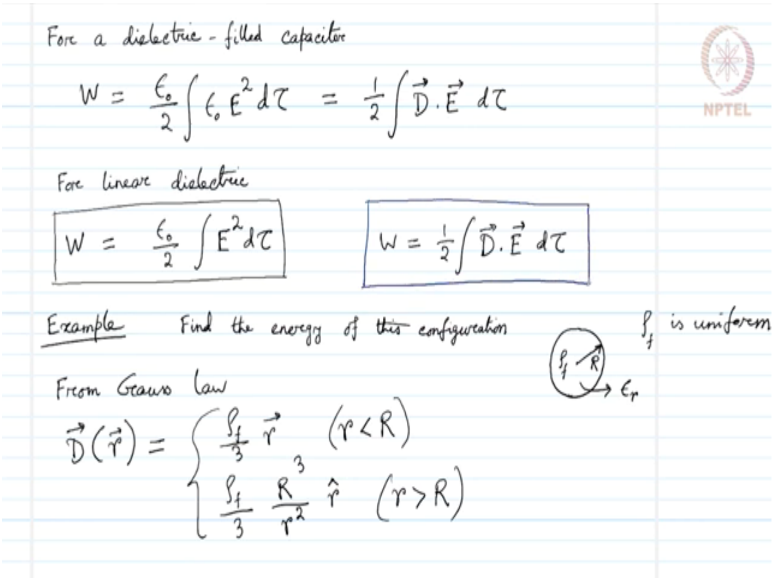
Now, let us consider the Energy in a Dielectric Material. How much work is done to store the amount of charge q and maintain a potential difference V in between the plates in a capacitor that is the energy stored in a dielectric material. So, it is the amount of energy supplied to charge a capacitor that is the energy stored in the dielectric material.

So, how do we find that, let us consider the differential work done to charge in the process of charging a capacitor and that will be the potential times the infinitesimal amount of charge and the potential if we have a q charge in the capacitor already that is q over C is the potential times d q that is the infinitesimal charge that we bring in that is nothing, but V d q. So, the work done to fully charge a capacitor is given as if there is capital Q charge finally, after fully charging it on each plate then it will be q over c d q.

And if we perform this integral we will find this to be half Q squared over C and from the definition of capacitance, we can write this as half CV squared. So, without knowing the total charge that we have put in the capacitor if we know the capacitance and if we can measure the potential difference between the plates that is the charging potential actually that is the maximum potential, it can asymptotically reach after getting fully charged.

We can still find the energy stored in the capacitor and that energy is nothing, but half times the capacitance times the potential difference squared; this is the amount of energy stored in the capacitor. And let us discuss this in comparison with the electrostatic energy stored in the electric field. So, we have seen that the capacitance in the presence of a dielectric material with relative permittivity epsilon r is epsilon r times it is capacitance if there was vacuum in between the plates. And the electrostatic energy in case of electric field is given by epsilon naught over 2 integration over the relevant space E squared d tau.

(Refer Slide Time: 04:10)



Now, if we have a dielectric field capacitor for that we have derived the expression that in W can be written as half CV squared and in the form of it is electric field we can write it is epsilon naught over 2 integration over epsilon r E squared d tau and that equals half integral over the displacement vector dot E d tau.

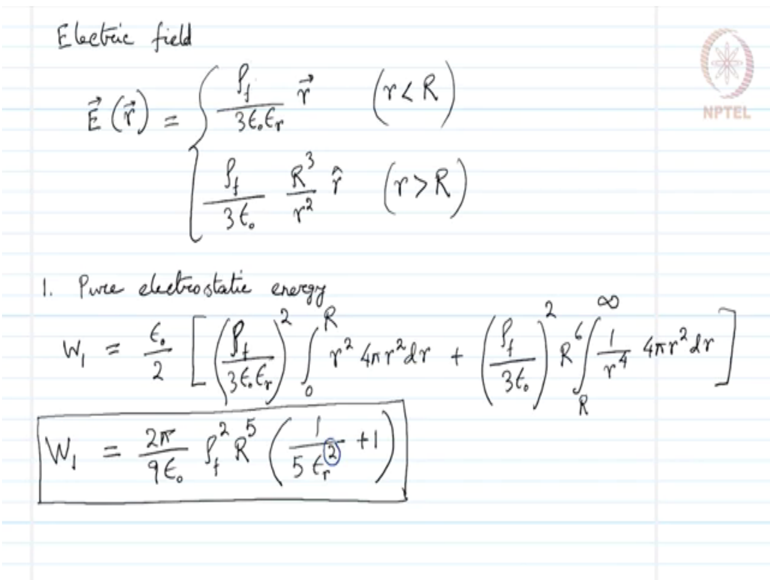
So, this is the general expression for energy stored in a dielectric material if the electric field E is developed and there is a displacement vector d. So, if we have a linear dielectric material, we have two expressions at hand that is W equals epsilon naught over 2 integration over E squared d tau and W equals half integration D dot E d tau. Let us see an example where that where these two expressions could apparently contradict each other.

We have a sphere of radius capital R filled with a dielectric material of dielectric constant epsilon r. We have this sphere the radius is capital R and it is filled with a material of dielectric relative permittivity epsilon r and it is uniformly embedded with free charge density rho f rho f is uniform and it is everywhere in the sphere. We have this kind of a situation. What is the energy of this configuration? We are supposed to find the energy of this configuration.

How do we do that? First we need to find the displacement vector as we have done earlier and we will have to apply Gauss law for that. The expression for displacement vector as a function of position vector would be given as rho f over 3 r vector and this is valid for r less than capital R that is inside the sphere and it would be rho f over 3 R cubed capital R cubed over small r squared and small r cap this is outside the sphere. This would be the expression for the displacement vector if we apply Gauss law and calculate it for this configuration.

Once we have the displacement vector, we can find the electric field.

(Refer Slide Time: 08:57)



So, the electric field would be given as this within the sphere and this outside the sphere. With this we know that the in this system the purely electrostatic energy that we can find is from the first expression that is integrating over the electric field squared. So, we will find the pure electrostatic energy from this expression, let us apply this expression first and see what we get.

We call it W 1 is given as epsilon naught over 2 rho f over 3 epsilon naught epsilon r this quantity squared times integration from 0 to capital R r squared 4 pi r squared d r plus for the field outside we will write rho f over 3 epsilon naught squared R power 6 integration over R to infinity everywhere outside the sphere 1 over r power smaller power 4 4 pi r squared d r. This is the electrostatic energy from the first expression. And that turns out to be twice pi over 9 epsilon naught rho f squared R power 5 radius of the sphere power 5 1 over 5 epsilon r squared plus 1.

Now we now this is the expression for work done using the first method, let us box it and let us move on to the second method. The second method is this we will take the dot product of the displacement vector and the electric field, integrate over it over all space and half it. That will give us the electrostatic that will give us the energy of the system; energy of basically the same system.

(Refer Slide Time: 13:01)

2. $$W_2 = \frac{1}{2}\left[\left(\frac{\rho_f}{3}\right)\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)\int_0^R r^2\, 4\pi r^2\, dr + \left(\frac{\rho_f R^3}{3}\right)\left(\frac{\rho_f R^3}{3\epsilon_0}\right)\int_R^\infty \frac{1}{r^4}\, 4\pi r^2\, dr\right]$$

$$\boxed{W_2 = \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r} + 1\right)}$$

$\epsilon_r > 1 \qquad W_1 < W_2$

So, let us take another page and that way we call it method 2 W 2 is given as half rho f over 3 rho f over 3 epsilon naught epsilon r. So, the 3 factors of d and e we have written here, then we integrate from 0 to capital R inside the sphere r squared 4 pi r squared d r plus outside this sphere, the expression would be rho f capital R cubed over 3 this will come from the pre factor of displacement vector rho f capital R cubed over 3 epsilon naught. This is the pre factor of the electric field, then integration everywhere outside the sphere capital R to infinity 1 over r power 4 pi r squared d r. And if we perform this integral we will obtain twice pi over 9 epsilon naught rho f squared r power 5 1 over 5 epsilon r plus 1.

So, this is the expression for W 2, let us box it and let us compare the two expressions that we have obtained. We have from the first method obtained this expression for W 1, the only difference with the second expression is this square here epsilon 1 over 5 epsilon r squared; in the second expression, we do not have any power of this epsilon r. Now we have epsilon r is greater than 1 it has to be greater than 1 for any value of susceptibility because it is 1 plus susceptibility and therefore, W 1 is less than W 2. What do we mean by that, how could work how could energy be different in different methods, that is not possible right? Let us analyze what is in W 2.

(Refer Slide Time: 16:32)

Analyze W_2

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\vec{r} & (r < r') \\[2ex] \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\dfrac{r'^3}{r^2}\hat{r} & (r' < r < R) \\[2ex] \dfrac{\rho_f}{3\epsilon_0}\dfrac{r'^3}{r^2}\hat{r} & (r > R) \end{cases}$$

$$dW = -dq\left[\int_{\infty}^{R}\vec{E}\cdot d\vec{l} + \int_{R}^{r'}\vec{E}\cdot d\vec{l}\right]$$

W 1 is the only electrostatic energy we have calculated from the electric field and we have already we have derived this earlier. So, we are confident about this, let us see what goes into W 2. We start with the uncharged, unpolarized dielectric sphere and bringing the free charge in infinitesimal installments of d q.

So, we have considered this sphere here and if we consider there is no charge in this and d q installment of charges brought in here and this process is repeated until the total charge of this sphere becomes Q. We consider something like this this kind of a process. And if we considered that the sphere is filled with charge layer by layer then what happens is, when we have reached radius r prime let us say this one. This one has radius r prime up to this, it is uniformly charged and outside it is not charged yet in our process of charging the sphere, if we consider this kind of a situation.

Then the electric field for this configuration would be rho f over 3 epsilon naught epsilon r r vector for r less than r prime. It would be rho f over 3 epsilon naught epsilon r r prime cubed over r squared r cap direction for r prime less than r less than capital R; that means, in between r prime and capital R that region. So, capital R is this much and the electric field outside the sphere would be rho f over 3 epsilon naught r prime cubed over r squared r cap outside the sphere; this would be the expression for the electric field.

Now, the work required to bring the next installment of charge d q and make another shell that amount of work can be written as d w equals minus d q. So, we are bringing in this charge from infinity integration over infinity to R capital R E dot d l plus integration R to r prime, we will fix that spherical shell at r prime radius E dot d l. This much work we have to do and let us perform this integral and find out what it is.

(Refer Slide Time: 20:26)

$$dW = -dq\left[\frac{\rho_f r'^3}{3\epsilon_0}\int_{\infty}^{R}\frac{1}{r^2}\,dr + \frac{\rho_f r'^3}{3\epsilon_0\epsilon_r}\int_{R}^{r'}\frac{1}{r^2}\,dr\right]$$

$$dW = \frac{\rho_f r'^3}{3\epsilon_0}\left[\frac{1}{R} + \frac{1}{\epsilon_r}\left(\frac{1}{r'} - \frac{1}{R}\right)\right]dq$$

$$dq = \rho_f\, 4\pi r'^2\, dr' \qquad r' = 0 \rightarrow r' = R$$

$$W = \frac{4\pi\rho_f^2}{3\epsilon_0}\left[\frac{1}{R}\left(1 - \frac{1}{\epsilon_r}\right)\int_0^R r'^5\, dr' + \frac{1}{\epsilon_r}\int_0^R r'^4\, dr'\right]$$

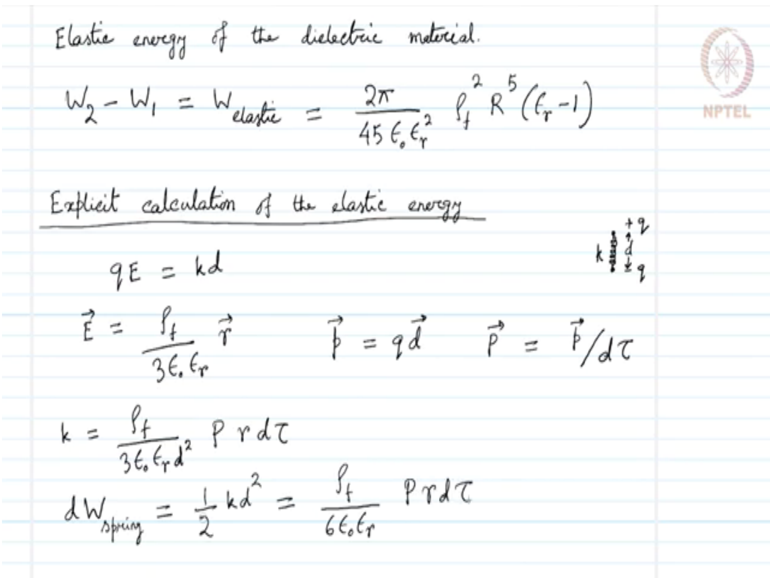
$$= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r} + 1\right) = W_2$$

So, this d w work is equal to minus d q rho f r prime cubed over 3 epsilon naught integration infinity to capital R 1 over r squared d r plus rho f r prime cubed over 3 epsilon naught epsilon r inside the material from capital R to r prime 1 over r squared d r. This is the this integration describes the process of bringing in the charge and with this consideration. So, simplifying this expression it becomes rho f r prime cubed over 3 epsilon naught 1 over capital R plus 1 over epsilon r 1 over r prime minus 1 over capital R times d q, this is the expression for d w.

And this will increase the radius r prime. So, we can write in terms of the increment of the radius r prime this as d q equals rho f times 4 pi r prime square d r prime and the total work is done in the process of going from r prime equals 0 to r prime equals R. So, if we put d r prime as and integrate over it then the limit of this integration will be from 0 to capital R. And then we can write the expression for work done to perform this process as 4 pi rho f squared over 3 epsilon naught. 1 over capital R 1 minus 1 over epsilon r integration from 0 to capital R r prime power 5 d r prime plus no, but this does not end here this is the factor for this integral plus 1 over epsilon r integration from 0 to capital R r prime power 4 d r prime. This is the total work done and that gives us twice pi over 9 epsilon naught rho f squared capital R power 5 1 over 5 epsilon r plus 1 and that equals W 2.

Then we have retrieved W 2 from this consideration while W 1 that we have calculated from the electric field that was the energy stored in the electric field. So, what energy is more in this situation, we have inferred that W 1 is less than W 2. So, W 2 has some more energy stored in the system and which energy is that, can we find out the source of this energy, this must be the elastic energy of the system of the dielectric material.

(Refer Slide Time: 25:34)



So, we have calculated in the first process the energy from the electric field. So, we have found how much energy is stored in the electric field, but this system under consideration is not only the electric field. We also have the dielectric material and the dielectric material can store some energy in itself, because it gets some polarization it develops it gets a little bit deformed elastically and that stores some energy in it and the amount of elastic energy would just be W 2 minus W 1 in this case.

So, we can write that W 2 minus W 1 is the elastic energy which is twice pi over 45 epsilon naught epsilon r squared rho f squared R power 5 epsilon r minus 1. So, we have just said this can we explicitly calculate the elastic energy stored in the material, let us try doing that. So, we can assume that the dielectric material is a collection of tiny dipoles each comprising plus 1 and minus 1 charge and a spring constant k is attached to that; that means, we are assuming this kind of a situation this is plus q charge.

And this is minus q charge and a string is attached to this, where the spring constant is given as k. We can assume this kind of a system and a collection of this representing the dielectric material. In the absence of any electric field, the positive and negative end of these two charges so, they coincide they fall on each other. And if we assume that one end of each dipole is nailed in it is position; let us say this negative charge is fixed it cannot move only the positive charge can move.

Or we can assume the other way around that the positive charge is nailed it cannot move like the nucleus of an atom only the negative charge cloud can move like the electron cloud of an atom. If we have that kind of a situation then and the other end is free to respond to the applied electric field. If we have this kind of a of an understanding then in the presence of an electric field the electric force and the spring force would balance each other.

So, the spring will get elongated to such an extent where the external electric field applied on this is exactly balanced by this force developed due to the spring. And in that situation we can write if the distance between these two charges become d at equilibrium, we can write that the by force balancing q E equals k d, then the electric field vector can be represented as rho f over 3 epsilon naught epsilon r r vector.

And the dipole moment for this kind of a tiny dipole is given by q times the d vector this mean separation vector and the polarization P can be given as this dipole moment vector over the tiny volume d tau that we have under consideration. Thus we will have k equals rho f over 3 epsilon naught epsilon r d squared times the polarization magnitude r d tau. That is the spring constant infinitesimal work done on the string in the process of displacement is half k d squared. And that equals rho f divided by 6 epsilon naught epsilon r times the polarization r d tau.

(Refer Slide Time: 32:08)

Total energy of all the springs

$$W_{spring} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\int P\, r\, d\tau$$

For a linear dielectric $\vec{P} = \epsilon_0 \chi_e \vec{E}$

$$= \epsilon_0 \chi_e \frac{\rho_f}{3\epsilon_0\epsilon_r}\vec{r} = \frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r}\vec{r}$$

$$W_{springs} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\,\frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r}\,4\pi\int_0^R r^4\, dr$$

$$= \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5(\epsilon_r - 1) = W_{elastic} = W_{spring}$$

So, the total energy of total energy in the material coming from all the springs that can be given as rho f over 6 epsilon naught epsilon r integration P r d tau. For a linear dielectric we know that the polarization can be expressed as the polarization vector is epsilon naught chi E times the electric field which is nothing but epsilon naught chi e rho f over 3 epsilon naught epsilon r r vector, that equals epsilon r minus 1 times rho f over 3 epsilon r times r vector.

Now if we work out the energy due to all the springs with this expression of polarization we will find this is equal to rho f over 6 epsilon naught epsilon r epsilon r minus 1 times rho f over 3 epsilon r 4 pi times integration over 0 to capital R so, 4 pi comes from theta and phi integration. So, this system is symmetric against theta and phi we can just write 4 pi for that integral only integral over r we have to find out that is r power 4 d r.

And if we calculate this we will find twice pi over 45 epsilon naught epsilon r squared rho f squared R power 5 epsilon r minus 1 and exactly this was the expression for elastic work done that we were talking about earlier. So, this was the difference between W 2 and W 1 that we have calculated and we said that that was the elastic energy stored in the dielectric material itself.

So, by working out this simple model that we have two charge centers separated by a spring and they attract each other by the spring const because of the spring constant because of the spring being there, under the influence of electric field they gets sep. These two charges get separated and the spring wants them to bring closer together again.

And this there is this balance of force and that stores some energy in this material with this simple model we have found out the same expression for the spring energy collection of all springs in the material as the difference between W 2 and W 1 in our earlier work. That means, this elastic energy that we have said earlier must be elastic energy is same as the spring energy and we were correct in understanding that the difference between W 2 and W 1 was the spring energy and nothing else.