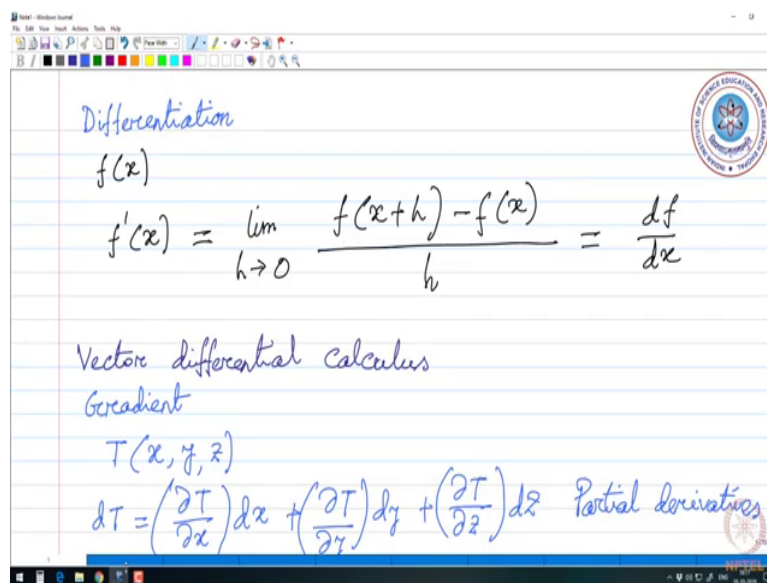


Electromagnetism
Dr. Nirmal Ganguli
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Lecture - 04
Vector differential calculus: Gradient

Hello everybody. In the last discussion we have learned vector algebra and now it is about Vector calculus. So, we will start with differential calculus; obviously, and before going to vector calculus we would first see how ordinary derivatives are calculated with a scalar function.

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The image shows a digital whiteboard with handwritten mathematical notes. At the top, it says "Differentiation" and "f(x)". Below that is the definition of the derivative: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{df}{dx}$. Further down, it says "Vector differential calculus" and "Gradient". Below that is the formula for the total differential of a scalar field T(x, y, z): $dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$, with the text "Partial derivatives" written to the right of the formula. The whiteboard also features a logo of the Indian Institute of Science Education and Research, Bhopal, in the top right corner.

So, let us consider differentiation first, if we have a function $f(x)$ then its derivative first order derivative with respect to x that is $f'(x)$ is defined as limit h tends to 0 $f(x+h) - f(x)$

$\frac{dx}{h}$. We are familiar with this definition and this in the shorthand notation is written as df .

Geometrically we interpret this derivative as the slope of the function. Now, if we are interested in vector differential calculus then the simplest operation would be gradient of a scalar field that becomes vector. How do we define gradient? Let us consider we have a scalar field T and that is a function of x , y and z in 3 dimensional Cartesian coordinate system. So, we can write dT as $\frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$ in the form of partial derivatives..

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The image shows a handwritten derivation on lined paper. At the top, the differential dT is expressed as a dot product of a vector and a differential displacement vector $d\vec{l}$. The vector components are the partial derivatives of T with respect to x , y , and z , multiplied by unit vectors \hat{i} , \hat{j} , and \hat{k} respectively. The differential displacement vector $d\vec{l}$ is shown as $(dx\hat{i} + dy\hat{j} + dz\hat{k})$. A blue arrow points from the vector in the first equation to the second equation, which defines the gradient vector $\vec{\nabla} T(x, y, z)$. The third equation shows the explicit form of the gradient vector as the sum of the partial derivatives multiplied by their respective unit vectors. The word "Gradient" is written below the final expression. A circular logo of the National Institute of Technology is visible in the top right corner of the paper.

$$dT = \left(\frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j} + \frac{\partial T}{\partial z} \hat{k} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \vec{\nabla} T(x, y, z) \cdot d\vec{l}$$

$$\vec{\nabla} T = \left(\frac{\partial T}{\partial x} \right) \hat{i} + \left(\frac{\partial T}{\partial y} \right) \hat{j} + \left(\frac{\partial T}{\partial z} \right) \hat{k}$$

Gradient

Now, if we make a grid we want to take the gradient of this then the gradient can be so, dT can be written in the form $\frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$. This quantity dotted with $dx\hat{i} + dy\hat{j} + dz\hat{k}$. And we know that this right hand

side is nothing, but the differential line element; that means, this quantity becomes the gradient of the scalar field T that is a function of x , y and z dotted with the line element dl .

Now, this gradient of T is nothing, but this quantity here. So, here we define gradient of T written in this way nabla followed by T is $\text{del } T = \text{del } x \hat{x} + \text{del } y \hat{y} + \text{del } z \hat{z}$. Geometrically gradient means the direction in which the scalar field is changing. So, gradient of a scalar field is a vector quantity that has a magnitude and a direction, it is very important to note.

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The image shows a handwritten derivation on a digital whiteboard. The text reads: "Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ ". Below this, the gradient vector is calculated as follows:
$$\vec{\nabla} r = \frac{\partial r}{\partial x} \hat{x} + \frac{\partial r}{\partial y} \hat{y} + \frac{\partial r}{\partial z} \hat{z}$$

$$= \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2 + z^2}} \hat{x} + \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2 + z^2}} \hat{y} + \frac{1}{2} \frac{2z}{\sqrt{x^2 + y^2 + z^2}} \hat{z}$$

$$= \frac{x \hat{x} + y \hat{y} + z \hat{z}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\vec{r}}{r} = \hat{r}$$
The whiteboard also features a logo of the Ministry of National Education in the top right corner and a Windows taskbar at the bottom.

Let us consider one example of calculating a gradient, let us try finding the gradient of the position vector not really vector the magnitude of the position vector r in 3 dimensional Cartesian coordinate system. As we know the definition of gradient is this is given as $\text{del } r = \text{del } x \hat{x} + \text{del } y \hat{y} + \text{del } z \hat{z}$

$x \hat{i} + y \hat{j} + z \hat{k}$. And, with the definition of r that is given here we can clearly see that the derivative of r with respect to x will bring us half in front.

And, because there is x^2 we will have $2x$ here and it will the power of r would be minus half; that means, 1 over $x^2 + y^2 + z^2$ square root of this quantity \hat{i} . Similarly, for derivative with respect to y it will be half $2y$ over $x^2 + y^2 + z^2$ square root of that \hat{j} and for z it would be half $2z$ over the same thing which is the half and two these things will cancel. And, we will be left with $x \hat{i} + y \hat{j} + z \hat{k}$ over $x^2 + y^2 + z^2$ square root of this quantity.

So, the numerator is nothing, but the r vector over the denominator the magnitude of r ; that means, the gradient of the scalar of the scalar magnitude of the position vector gives us the direction of r , that is unit vector along the position vector r . Let us consider another example.

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Example

$$f(x, y, z) = x^2 + y^3 + z^4$$

$$\vec{\nabla} f = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^3 + z^4)$$

$$= \hat{i} 2x + \hat{j} 3y^2 + \hat{k} 4z^3$$

$$= 2x \hat{i} + 3y^2 \hat{j} + 4z^3 \hat{k}$$

Let us consider a scalar field, if that is a function of all three position coordinates x , y and z is given as x squared plus y cubed plus z power 4. Now, if we calculate the gradient of this scalar field we will have we can write it as x cap del del x plus y cap del del y plus z cap del del z . This is the gradient operator and this operator is operated on the scalar field that is x square plus y cubed plus z power 4. So, we will get x cap 2 x plus y cap 3 y squared plus z cap 4 z cubed which is rearranging it properly twice x x cap plus 3 y squared y cap plus 4 z cubed z cap.

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Del operator

$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

So, we can see that we have developed a del operator out of this. The way in the previous example we have written the gradient of operator was this operator was equal to x cap del del x plus y cap del del y plus z cap del del z . Although this is a differential operator, it looks pretty much like a vector and just because it looks like a vector we can define its product with a scalar, its product with another vectors that will look like a dot product or a cross product.

But, we have to remember that it is not really a dot product and cross product of vectors rather it is a differential operator, we have to be careful about that.