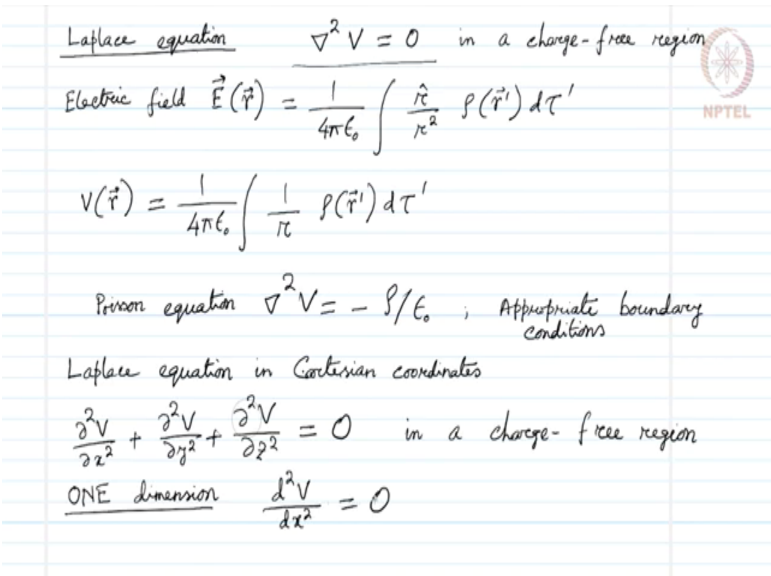
Electromagnetism
Dr. Nirmal Ganguli
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Lecture - 39
Laplace equation

We have already derived Laplace Equation in the context of electrostatic potential. Let us see what interesting we can learn from Laplace equation considering Laplace equation; because we did not already familiarize ourselves with solution of difficult differential equations. We will not go for a formal solution to Laplace equation; but we can still without performing a formal solution to the Laplace equation, get many interesting conclusions based on the Laplace equation. We will try highlighting few of those.

(Refer Slide Time: 01:12)

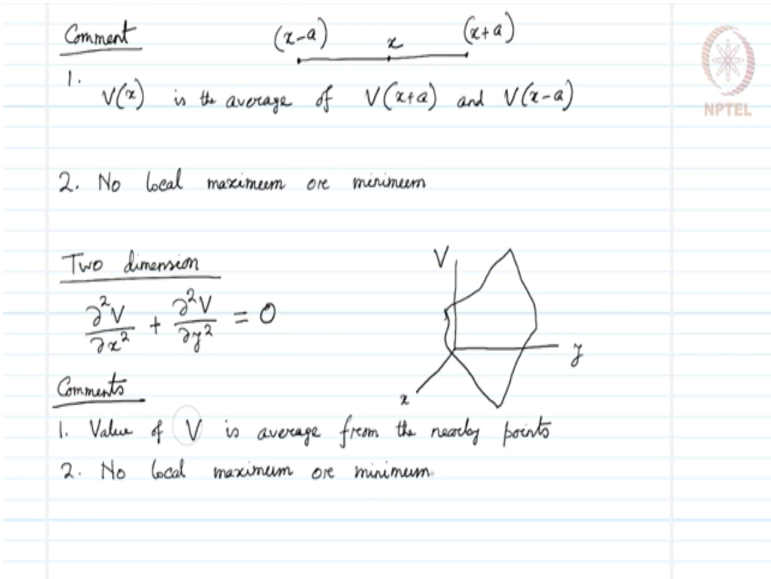


So, let us write down Laplace equation; Laplace equation is del square V equals 0 in a charge free region. If we consider an electric field E as a function of the position vector r, that can be expressed as 1 over 4 pi epsilon naught; integration r cap over r squared rho r prime d tau prime for a volume charge density. And, if we try to calculate the potential right from the charge density, right from the charge distribution; we can do that as we have already learnt by V equals 1 over 4 pi epsilon naught integration over 1 over r rho r prime d tau prime, this way we can find this. Now, even this integral may be too difficult to and handle it analytically. Now, if we have Laplace equation that is this equation at hand, in a source free source charge free region or say Poisson equation; that is del square of V equals minus rho over epsilon naught.

If we have these two at hands, any of these and appropriate boundary conditions; we can solve many interesting problems and many problems that are much more difficult than whatever symmetry we have discussed so far. Let us write the Laplace equation in Cartesian coordinate system, we can write del 2 V del x 2 plus del 2 V del y 2 plus del 2 V del z 2 going to 0 in a charge free region. If we have this, then let us consider for simplicity, the case of one dimension. In one dimension, we can write that d 2 V d x 2 equals 0, ok.

By looking at this, we can although with a set of appropriate boundary conditions, we can solve this; but we would not go in that direction, we would rather observe this equation and try to find out some properties of the solution to this equation.

(Refer Slide Time: 05:39)



We would like to comment on the properties of the solution to this Laplace equation. One comment we can write is that, if we consider a point a line like this, where in between this point has a coordinate x and this length is a on this side as well as on left hand side as well as right hand side; so this points coordinate becomes x minus a. And this points coordinate becomes x plus a; then for the solution that is V x at the midpoint here is nothing but the average of V x plus a and V x minus a; because there is no charge in between.

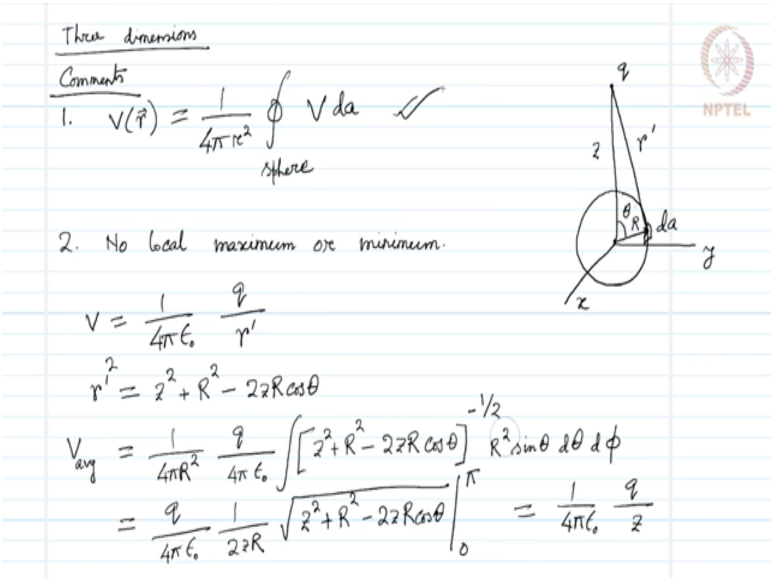
For and this is valid for any arbitrary a as long as there is no charge on this segment that we are considering. And then Laplace equations solution cannot have any local maximum or minimum. Within our region of interest, we do not have any free charge, any charge in that region; therefore, the solution to the Laplace equation solution cannot have local maximum or minimum within that region.

We can move to two dimension now. In case of two dimension, we can write Laplace equation as del 2 V del x 2 plus del 2 V del y 2 equals 0 and this is no longer an ordinary differential equation, we get a partial differential equation in this form. So, we can consider in the Cartesian coordinate system x, y, z a surface on this system something say like this; this is the surface on this coordinate system, and the potential can be a continuous surface in x, y, v space.

So, this surface can, if we consider x, y and along the z direction, we have plotted the potential, this is two dimensional case. So, this surface that we have drawn here, that could well represent the potential without any local maximum or minimum in this case.

So, let us also make comments in case of two dimensional situation; the value of the potential at any point is the average of those around that point. Value of V is the average from the nearby points. And the other comment that we can make is, it has no local maximum or minimum.

(Refer Slide Time: 10:26)



Let us move on to the case of three dimensions. In case of three dimension, we can write that, the value of the potential is the average from the points nearby; that means, the potential at point r with position vector r can be given as 1 over 4 pi r squared surface integral over a sphere V d a; where this sphere that we are considering is has no source charge in it. And then the other comment would still be the same that there would be no local maximum or minimum in the absence of a free charge.

Let us consider the average of the potential over a spherical surface of radius r, let us try to show this thing explicitly. Here we have a sphere, here is the center of the sphere; let us consider this as capital R, and here is the surface element that we are considering. Let us consider a point charge q here outside the sphere, so Laplace equation is valid in this sphere. And the distance from this point charge to the center of the sphere is z and we denote this distance as r prime. Let us draw the x, y and z coordinates; this is x, this is y, this is theta in spherical coordinate system and this element here is d a.

If we have this, then the potential due to this point charge q at the surface element here, can be given as V equals 1 over 4 pi epsilon naught q over r prime sorry, r prime this way; because this the distance from q to the surface element is r prime. And r prime can be given as looking at the geometry of this picture z squared plus R squared minus 2 z R cosine of theta, this is the same triangular rule. And then V average the average potential is 1 over 4 pi R squared q over 4 pi epsilon naught, integration z square plus R squared minus 2 z R cosine of theta square root of this quantity, here I had a square, I missed it, square root of this quantity that is power it is 1 over r. So, it is 1 over square root of this quantity powered minus half. And we have 2 multiply it with the surface element that is R square d sorry, R squared sine theta d theta d phi. And if we evaluate this integral, we will find this equals q over 4 pi epsilon naught; 1 over 2 z R z square plus R squared minus 2 z R cosine of theta square root of this, and the limit goes from 0 to pi that is the limit over theta. And then it becomes 1 over 4 pi epsilon naught q over z, this becomes the average of the potential.

Now, this is precisely the potential due to this point charge q here at a z at a distance z from the center of this sphere. So, if we average the potential over the surface of the sphere, we get the potential at the center of the sphere. So, in this example, we have verified the statement that we made here. So, we understand that our comment on Laplace equation is indeed valid.