Electromagnetism
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Lecture – 22
Tutorial on vector calculus and curvilinear coordinates

Hello, now we are going to start our second tutorial. The second tutorial is about integral calculus with vectors. And integral calculus is not as straightforward as the differential calculus with vectors. So, we need far more attention in this tutorial and also solving subsequent problems after working out this tutorial. Let us see what kind of problems we handle in the integral calculus with vectors.

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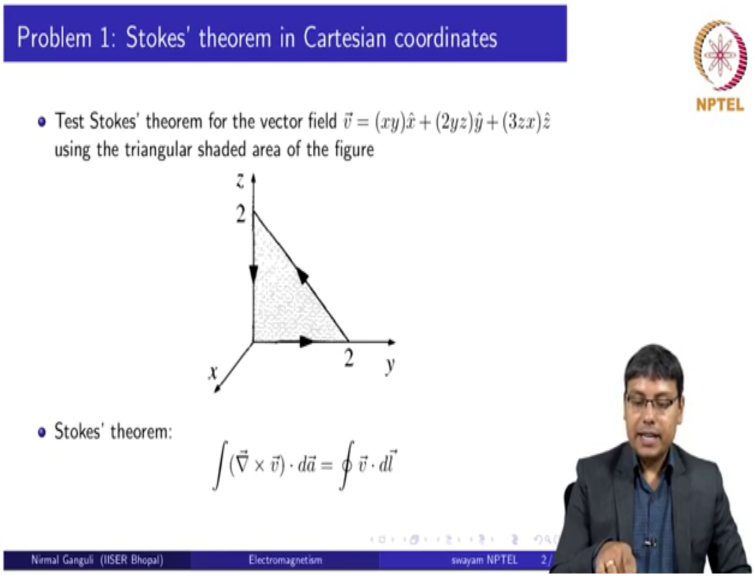

So, we know the Stokes' theorem, Stokes' theorem tells us that the curl of a vector quantity, if we perform a surface integral of that, that is equal to the vector quantity without taking a curl integrated over a closed line that encloses the surface that is Stokes' theorem. And we have been given a vector field v equals xy x cap plus 2 y z y cap plus 3 z x z cap. And we have been given a shaded region in this picture, we have xyz Cartesian coordinate system right handed Cartesian coordinate system. And in this we have a triangular region on a one y z plane, and we are supposed to verify Stokes' theorem on this surface, so, on this surface, as well as the lines that we have drawn using arrows. So, the line elements will go along that arrow.

So, what would be the direction of this surface? If we consider these line elements then a cross product will tell us that the surface direction will be out of the screen. So, assuming the direction of this area element out of the screen and the direction of the line elements are as shown by the arrows, we will try to verify Stokes' theorem. So, you can pause the video now, and try verify it yourself and then restart the video. So, what is Stokes' theorem? A Stokes' theorem is the surface integral of curl of v dot da that is equal to closed line integral over v dot dl.

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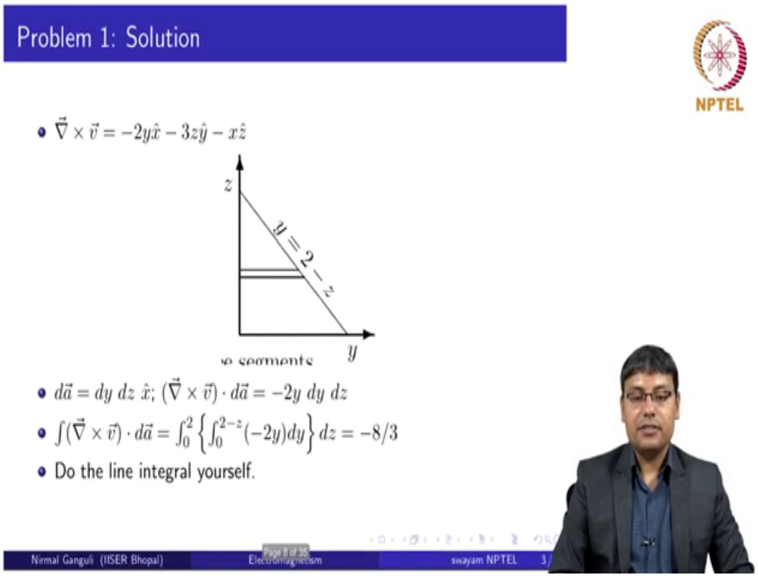

We are supposed to verify this theorem. And how do we verify this? We first calculate the curl of this vector field and for the given vector field the curl will be minus 2 y x cap minus 3 z y cap minus x z cap. And now we can see that we can consider an area element like the one shown here in the picture, so the area element is by shown by these two parallel lines here, it is a parallelogram that we have. No, it is not a parallelogram, it is a trapezoid that we have here. And if we move this trapezoid along z axis, we can cover the entire area of this triangle. The line the slanted line that we have here connecting z and y-axis the equation of that line is y equals 2 minus z. With that the expression for this area element we can write down as dy dz x cap, x cap is outside the screen above the screen.

And curl of the vector dot da. So, da is along x cap only x component of the curl will matter here. So, the curl of the vector dot da would be minus 2 y dy dz. And we are supposed to integrate over this within appropriate range of y and z. So, what would be the range of y, the range of y as we can see will be from 0 to 2 minus z that is from 0 to this line this slanted line that we have. And the range of z would be from 0 to 2. So, for performing this integral, we will have to first perform an integral over y. So, the argument will be minus 2 y dy and the limit of this integral will be from 0 to 2 minus z, and then whatever we get after performing this integral, we perform the z integral on this over the range 0 to 2. And by performing these two integrals, we will obtain the value minus 8 over 3. Please perform this integral yourself.

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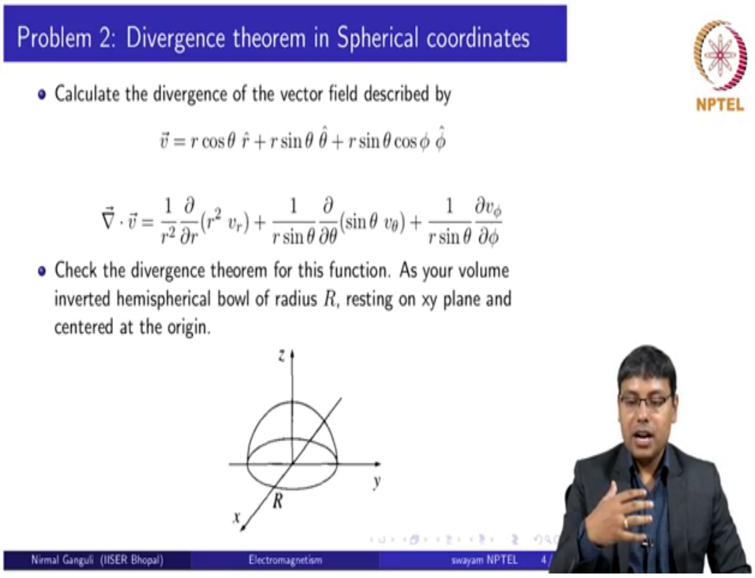

And we have to see the same thing using the line integral as well. So, how do we do that, we consider the lines that are shown here using the arrows, take the line elements accordingly. So, the line element along z-axis in the direction given is minus dz z cap, along y-axis it would be dy y cap. And along the slanted line, we have to find this out it will that line element we can find out by differentiating the equation of the line we have dy equals minus dz there. And from that we can find this line element and putting appropriate limits, we can perform the line integral. Please perform the line integral yourself and try to verify whether Stokes' theorem is valid for this system or not. Obviously, the answer would be that the Stokes' theorem is valid it is in general valid for any kind of situation. So, here it should also be valid. Please verify that yourself.

Now, let us consider the second problem of this tutorial. The second problem wants us to verify the divergence theorem in spherical coordinate. So, we are given a vector field v given as r cosine theta along the direction r cap that is the r component of the vector the theta component is r sine theta that is along the direction theta cap plus the phi component is r sine theta cosine phi that is along the direction phi cap, this is the vector field that we have.

And we are supposed to verify the divergence theorem. And the divergence of a vector field v in spherical coordinate system, this is the expression for that we have obtained this earlier we have written this down earlier is 1 over r squared del del r r squared the r component of the vector plus 1 over r sine theta del del theta sine theta times the theta component of the vector plus 1 over r sine theta del del phi of the phi component of the vector.

Now, this problem asks you to check the divergence theorem for this the given vector field here. And you have a inverted hemispherical bowl of radius capital R resting on xy plane and cantered at the origin like the picture here. So, it is you can consider is it as a northern hemisphere only that thing we have. So, the surface here is a curved surface and a flat surface for the surface integral part of the divergence theorem, and the volume is that of a hemisphere only the northern hemisphere in this case.

So, what how do we obtain northern hemisphere in terms of spherical coordinate? We can have r small r going from 0 to capital R that gives us the range of r; theta will range from 0 to pi by 2. If we go to pi that will give us the entire sphere for the northern hemisphere, we will have it from 0 to pi by 2; for southern hemisphere it will be pi by 2 to pi. So, this is a northern hemisphere we will have from 0 to pi by 2, and phi will span the entire region that is 0 to 2 pi. Now, you can pause the video and verify the theorem, try to verify the theorem yourself the divergence theorem. After doing that you can replay the video and see how we can actually do it ok.

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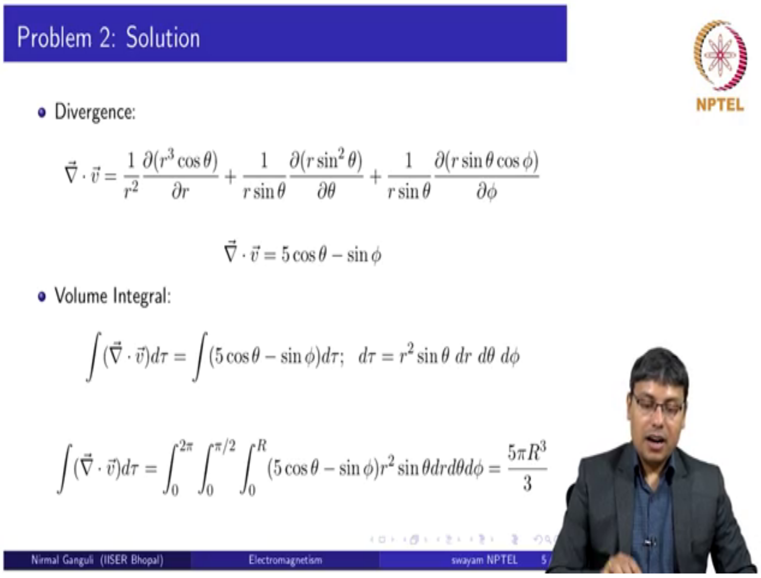

So, if we perform if we calculate the divergence of the given vector, the expression would be 1 over r square del del r of r cubed cosine of theta plus 1 over r sine theta del del theta of r sine square theta plus 1 over r sine theta del del phi of r sine theta cos phi. So, the vector the divergence of the vector turns out to be 5 cosine theta minus sine phi. And if we perform a volume integral over this divergence, so we will have to multiply it with the volume element. And the volume element in this context is r square sine theta drd theta d phi. We have already talked about the ranges of r theta and phi.

So, if we put the appropriate ranges r integral over r the range is from 0 to capital R integral over theta the ranges from 0 to pi by 2, the integral over phi the ranges from 0 to 2 pi, the integrand is 5 cosine theta minus sine phi multiplied by the volume element r squared sine theta drd theta d phi. If we perform this integral, please perform this integral yourself, you will find 5 pi capital R cubed over 3. So, this is the volume integral of the divergence.

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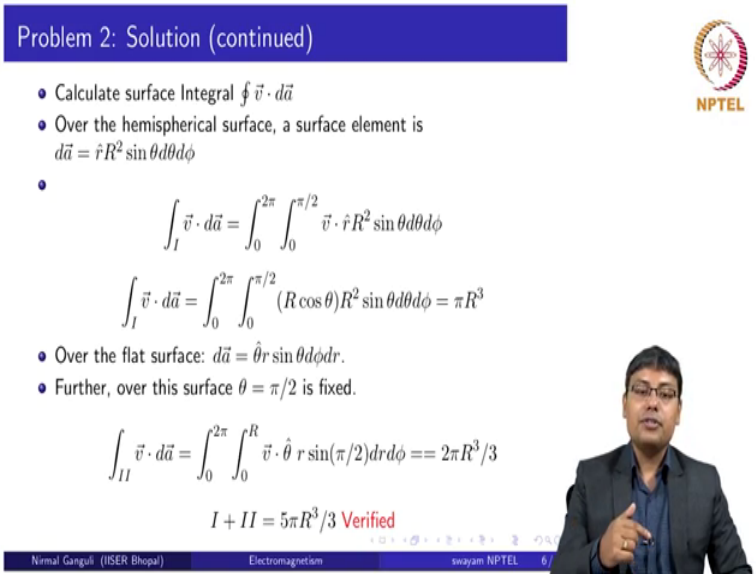

Now, we have to perform the surface integral v dot da over the surface that encloses this volume. So, which surface encloses this volume according to this picture, the curved surface of the spherical curved surface plus the flat surface that is lying on the xy plane here. So, we will have to consider both surfaces. How do we do that? Over the hemispherical surface we can take the surface element da vector that is given as arc that is along r cap direction perpendicular to the surface, and the magnitude of that element is r squared sine theta d theta d phi that is something we are already familiar with.

So, over that hemispherical surface curved surface, the surface integral v dot da becomes the range of phi is from 0 to 2 pi range of theta is from 0 to pi by 2, we have v dot r cap capital R squared sine theta d theta d phi. So, we will only have the r component of the vector relevant here, everything will everything else will go to 0, because we are taking a dot product along with the r cap direction. So, we will have integration over capital r cosine theta capital R because at this surface on this surface the value of small r will become capital R times r square sine theta d theta d phi and if you perform this integral you will find pi times capital R cubed as the value of this integral and we are left with the flat surface now.

On the flat surface d a will be along theta cap direction, theta cap direction means perpendicularly below minus the direction on that xy plane. Theta cap direction changes its direction depending on what value of theta you are considering. For theta equals pi by two the direction is minus z direction, and that is the direction of this surface. We always consider outward direction as the direction of the surface. So, its theta cap r sine theta d phi d r that is the surface element. Now, remember that r is a variable in this case small r is a variable that has range from 0 to capital R theta is fixed that is pi by 2 and phi has a range from 0 to 2 pi. So, we will perform this integral for over this surface theta is pi by 2. So, we have a fixed value of sine theta that sine pi by 2. So, the second integral over the flat surface v dot da becomes integration over phi from 0 to pi by 2 integration over r from 0 to capital R v dot theta cap r sine pi by 2 d rd phi.

So, only the theta component of the vector will be relevant here and that will after performing this integral, you will find this equals twice pi r cubed over 3. Now, if you add the first and the second integral, you will find the total is 5 pi r cubed over 3 and that is what we found from the divergence by performing the volume integral over the divergence. So, the divergence theorem is verified for the given vector in spherical coordinate system.

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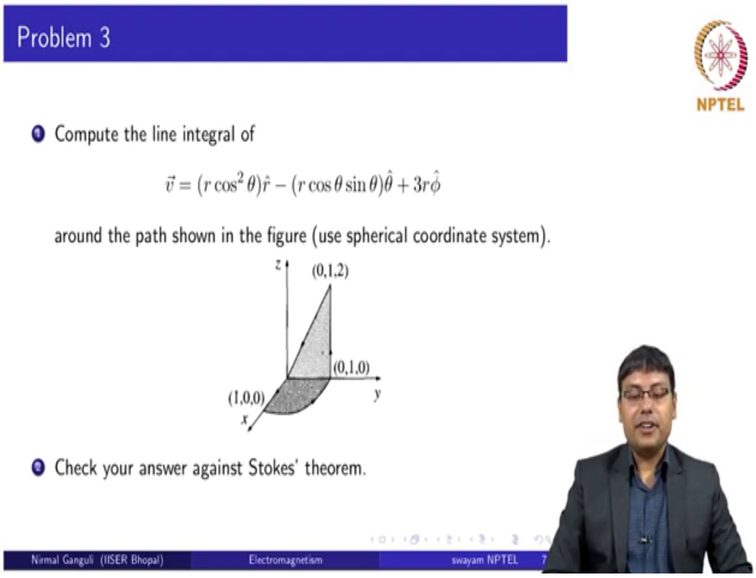

Now, let us consider a third problem in this tutorial. You are given a vector v in spherical coordinate system, it is given as r cos squared theta along r cap direction minus r cos theta sine theta along theta cap direction plus 3 r along phi cap direction. Now, you are asked to compute the line integral of this vector along the path shown in the figure. So, in this figure, we have a triangular part in the y z plane and a quadrant of a circle in the xy plane. This is the path that the direction is given here. You are supposed to use spherical coordinate system although the path is shown in Cartesian coordinate system, and the coordinates that are given for different points in this diagram are also in Cartesian coordinate system.

Because the vector field is given in spherical coordinate system, it will be much easier to convert these coordinates given in Cartesian coordinate system into spherical coordinate system and perform the line integral over the shown direction. And then check your answer against Stokes' theorem. What does Stokes' theorem tell us, Stokes' theorem tells us that this line integral that makes that encloses a surface a part of the surface is lying on a xy plane and another part is lying on y z plane.

So, this surface is enclosed by this line that is shown here, and the line integral of the vector along this line that is shown here would be equal to the surface integral on the shaded surface of the curl of this vector that is something we are supposed to verify. Now, please pause the video and try doing it yourself, and then restart the video later ok.

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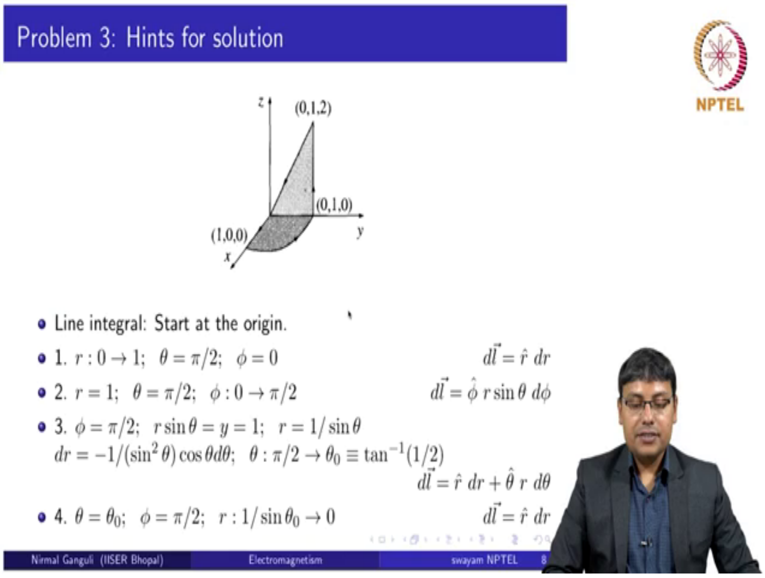

So, if we want to perform the line integral, we can start at the origin, and the first thing we note is that r on the first line segment is ranges from 0 to 1, theta is fixed that is pi by 2, and phi is 0. So, dl the line element is r cap d r. On the second that is the arc that we have on xy plane there r is fixed that is 1, theta is pi by 2 that is also fixed, and phi ranges from 0 to pi by 2 there. So, dl the line element is phi cap r sine theta d phi. And then on y z plane, the third line segment that is along the z-direction, we can have theta fixed at pi by 2, and r sine theta will be y that is 1. So, r will be 1 over sine theta and dr that is something interesting it will be minus 1 over sine square theta cos theta d theta.

Please find out how it becomes that this is if we have r given as 1 over sine theta then this would be the expression for d r theta would be pi by 2, 2 theta naught that would be the range of theta, theta naught is the value of theta at the top of that line. So, theta naught will be given as tan inverse of 1 over 2. And dl the line element along this line the vertically upwards lying that is a line along z direction would be given as r cap dr plus theta cap rd theta.

And finally, along the fourth line that is the slanted line coming back to the origin along that line theta equals theta naught the fixed value of theta that we have obtained earlier that is tan inverse of 1 over 2 phi equals fixed pi by 2 here and r that ranges from 1 over sine theta naught to 0.

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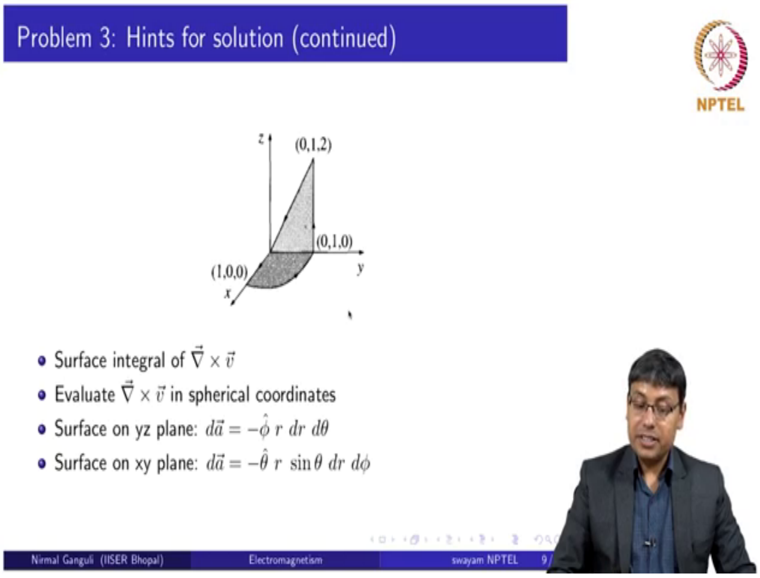

And well we have to find the dl the line element that is along r cap direction and dr, r cap dr is the line element along the slanted line. So, please perform this line integral and then we are supposed to perform this surface integral of the curl of this vector v. So, please calculate the curl of this vector.

After evaluating the curl of the vector v in spherical coordinate system, we need to perform this integral over two surfaces one surface is on y z plane. So, da would be minus phi cap that is above the screen that is the direction above the screen times rd rd phi. And the other surface is on xy plane and here the direction is along z-axis, so it is minus theta cap minus theta cap for xy plane points along z axis times r sine theta dr d phi these are the two surfaces, where we are supposed to perform the integral, surface integral of the curl of the vector. And please perform these integrals and verify the Stokes' theorem yourself that is the end of tutorial 2.