Electromagnetism
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Lecture – 19
Vector calculus in spherical coordinate system Part – 01

We have already discussed different curvilinear coordinates. And among the curvilinear coordinate systems the most important ones for this course are spherical and cylindrical coordinate system, because we will encounter systems with spherical and cylindrical symmetry.

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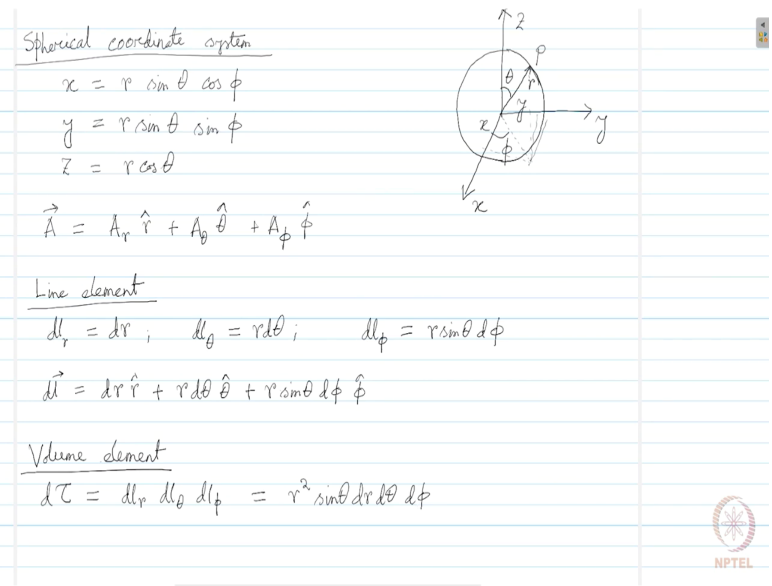

So, if we consider spherical coordinate system, we have already discussed earlier that x can be written as r sin theta cos phi, where r sin theta r theta and phi are defined over this kind of a sphere. This is the x-axis; this is the y-axis and this is the z-axis. If we have a point p here then the angle that position vector of point p makes with z axis is called theta, the position vector itself is called r. So, the distance from the origin to that point p is r. And if we project this onto the xy plane, we will find that it the projection makes an angle with the x-axis that angle is phi.

So, we can write down that x equals, x is this much length, y is this much length, sorry y is this much length; x equals r sin theta cos phi; y can be given as r sin theta sin phi, and z is given as r cos theta that you can clearly see from here. And a vector A in spherical coordinate system is represented as A r cap plus A theta theta cap plus A phi phi cap, where A r is this radial component A theta is the theta component, and A phi is the azimuthal component.

With this we have already found out its relation with Cartesian unit vectors, now let us consider the line element surface element and volume element for spherical coordinate system. A line element along r, let us write it down as dl with subscript r that is nothing but dr. If we consider the theta component of the line element, that would be r d theta which is clear from this picture. If you move r in this direction to increase the value of theta, so the distance it will travel is rd theta – r times the change in theta. Similarly, dl the phi component of the line element can be given as r sin theta d phi.

So, we are projecting this on to xy plane, and then changing the value of phi that means we have the radius of that projection as r sin theta and the change in angle is d phi. So, along phi direction the change in the length is r sin theta phi. So, the line element vector can be written as d r r cap plus r d theta theta cap plus r sin theta d phi phi cap.

With this it is if we can easily find out the volume element the volume element, d tau can be given as a triple product of this or simply multiplying dl r component with dl theta component and dl phi component, which is given as r square sin theta dr d theta d phi.

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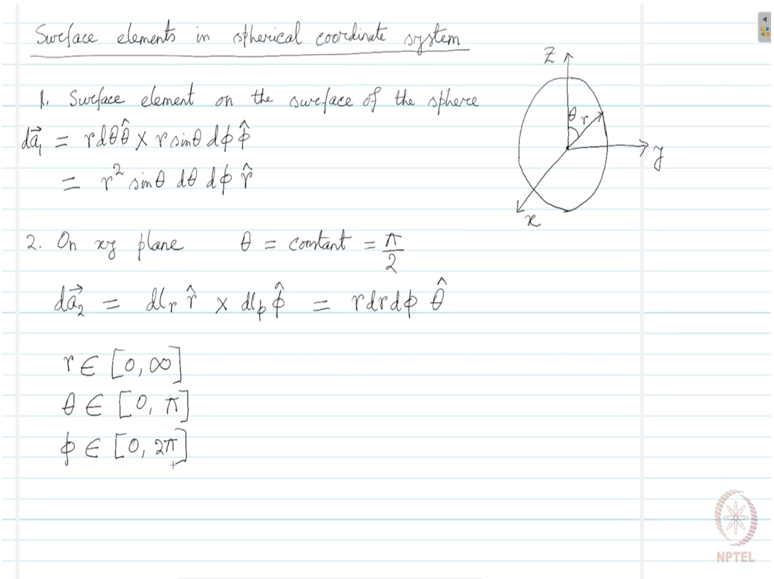

Now, let us come to the consideration of surface elements. In case of spherical coordinate system, we can certainly have different surfaces in here. And if we consider the surface of the sphere, the curved surface of the sphere, then this is the origin here is the z-axis from which we will measure theta. Then we can see that if this much is r, then rd theta will give us the distance along this direction, and we will have to multiply it with a perpendicular component. So, we will have to consider the phi component that is r sin theta d phi that will give us a component along the phi direction.

So, for a surface element on the surface of the sphere, we will have to consider, we will have to consider r as constant that is the time when we will the travel along the surface of the sphere. So, one component would be r d theta, and we will have to multiply it with the other component r sin theta d phi. Now, because surface element is a vector, we will have to consider theta cap direction here and phi cap direction here, and it will be a cross product.

Let us consider let us tell that this is the first kind of surface element vector, that means, it is given as r square sin theta d theta d phi and the cross product of theta cap and phi cap will give us r cap. This would be the first kind of surface element.

Now, if we consider the xy plane, if we are trying to find out a surface element on the xy plane, then what are the variables we can have r varying, we have theta fixed and phi can also be varying. So, on xy plane, we will have theta equals constant equals pi by 2. With this we will have da 2 the second kind of surface element as dl r r cap cross d l phi phi cap which is r dr d phi theta cap.

Please note once again here that the range of r is 0 to infinity, the range of theta is 0 to pi, and the range of phi is 0 to 2 pi for spherical coordinate system.

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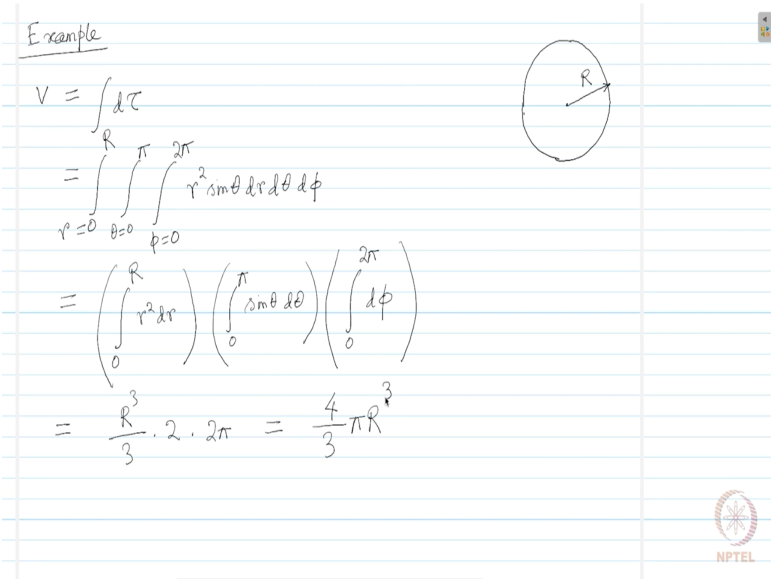

Now, let us consider an example. We want to find the volume of a sphere of radius capital R using the volume element that we have just found out. So, the radius of this sphere is capital R and we will perform a volume integral on the volume element that we have found out. So, the volume v would be given as integration over d tau. And here we will have to find the appropriate ranges for r theta and phi. We will have to perform three integrals 1 over r, 1 over theta and 1 over phi, and the integral will be performed over only d tau the volume element that is r square sin theta dr d theta d phi.

So, the range of r would be r is equals 0 to capital R, the range of theta would be theta equals 0 to pi, and the range of phi would be phi equals 0 to 2 pi. Now, we need to perform the integrals. And this integral would become integration over 0 to capital R r square dr times integration 0 to pi sin theta d theta times integration over 0 to 2 pi d phi, it is we can separate these integrals because none of the functions depend on each other.

For example, the function of r depends only on r not on theta or phi and similarly for theta and phi also there is no dependence on each other. Therefore, we can separate it this way and with that from r part we get R cubed over 3, from theta part we get 2, and from phi part we get 2 pi that gives us 4 over 3 pi capital R cubed. And this is the well known expression for volume that we all know.

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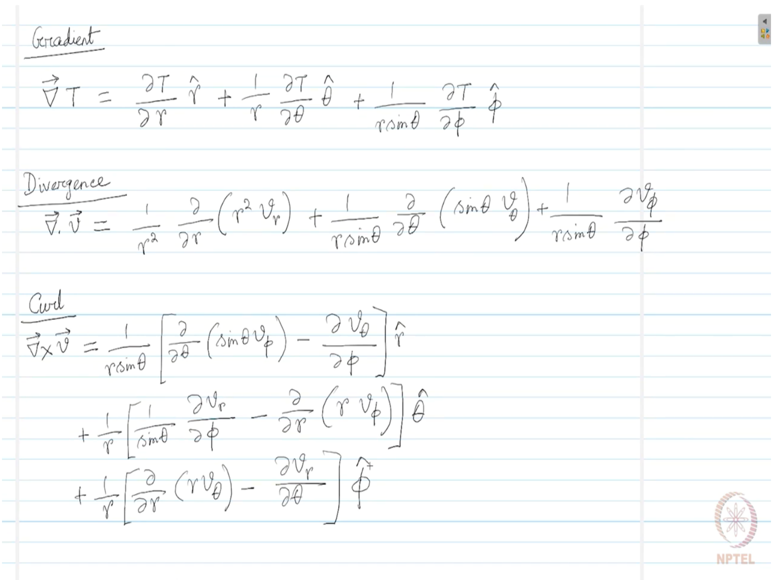

Now, let us consider gradient in spherical coordinate system. We will not derive the expression, you we have already discussed how we can derive it. So, we will just write down the expression for your reference. Gradient of a scalar field t is given in spherical coordinate as del t del r r cap plus 1 over r del t del theta theta cap plus 1 over r sin theta del t del phi phi cap.

We will give you the expression for divergence as well in spherical coordinate system. Divergence of a vector field v is given in this coordinate system as 1 over r squared del del r of r squared times the r component of the vector v r plus 1 over r sin theta del del theta sin theta times the theta component of the vector v theta plus 1 over r sin theta del v phi by del phi. This is the expression for divergence.

And finally, the expression for curl in spherical coordinate system can be given as curl of a vector field is expressed in spherical coordinate system as 1 over r sin theta del del theta sin theta v phi minus del v theta del phi r cap plus 1 over r 1 over sin theta del v r del phi minus del del r r v phi theta cap plus 1 over r del del r r v theta minus del v r del theta phi cap. This is the expression for curl.

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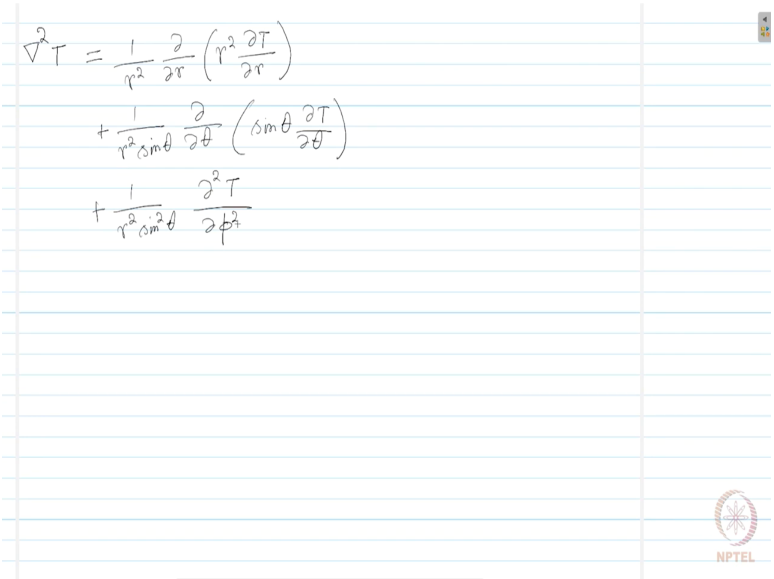

And after this we will have to give the expression for Laplacian that is del square in spherical coordinate system let us consider a scalar field T on which this del square is applying. So, we can write it as 1 over r squared del del r r squared del t del r plus 1 over r squared sin theta del del theta sin theta del t del phi sorry del theta plus 1 over r squared sin squared theta del 2 t del phi 2. This is the expression for the Laplacian operator del square in spherical coordinate system.