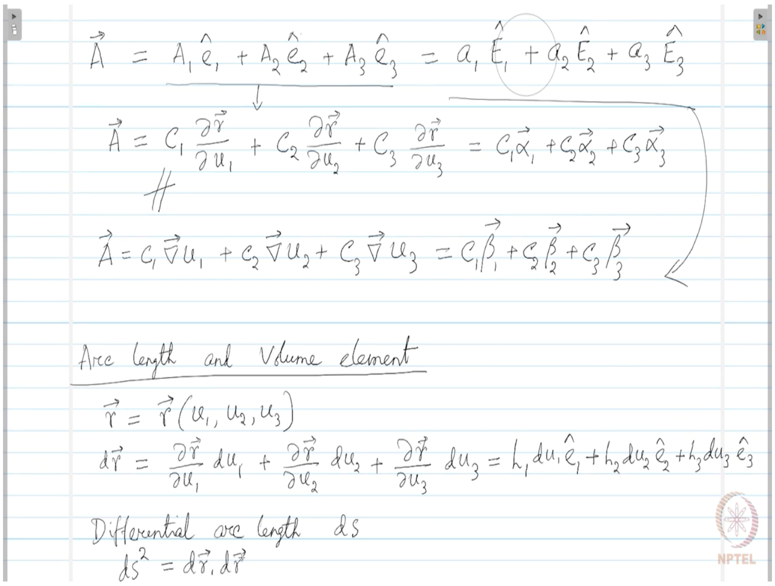
Electromagnetism
Dr. Nirmal Ganguli
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Lecture – 17
Differential vector calculus in curvilinear coordinate systems

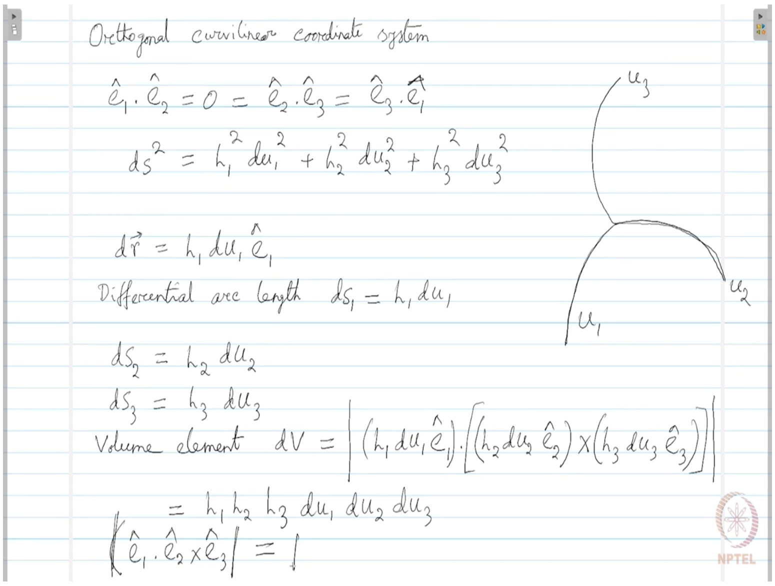
(Refer Slide Time: 00:32)



Now, let us try to find out the arc length and volume element in curvilinear coordinate system. Let us consider a position vector r is given as a function of u 1, u 2 and u 3 coordinates in that curvilinear coordinate system. Then we can write d r as in the first notation del r del u 1 d u 1 plus del r del u 2 d u 2 plus del r del u 3 d u 3 which is nothing but h 1 d u 1 e 1 unit vector plus h 2 d u 2 e 2 unit vector plus h 3 d u 3 e 3 unit vector.

Then the differential of arc length dS that can be determined as the arc length that is given as ds is nothing now we obtained ds square actually dS this quantity square would be dr dot dr.

(Refer Slide Time: 02:56)



So, we will obtain this quantity. And if we have an orthogonal curvilinear coordinate system, we will have the dot product of a unit vector width itself being 1, and the dot product with another unit vector will be 0. This will be the expression. Therefore, dS squared can be written as h 1 squared d u 1 squared plus h 2 squared d u 2 squared plus h 3 squared d u 3 squared this way, although this would not hold for non-orthogonal coordinate systems.

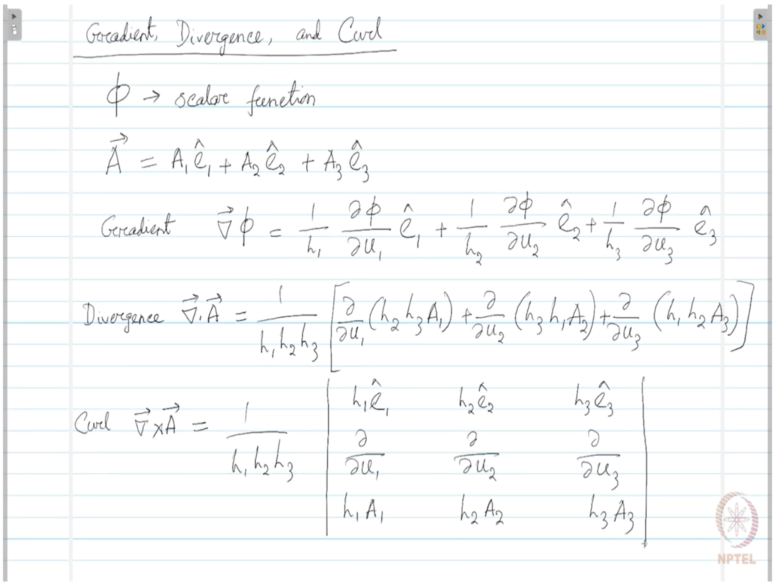
Now, along a u 1 curve u 2 and u 3 are constants. So, if we consider a curvilinear coordinate system where this is the u 1 axis, this is the u 2 axis and this is the u 3 axis something like this just a simple picture. And if we have along disc if we have an orthogonal curvilinear coordinate system, then along the curve of u 1 we will have the other components that is u 2 and u 3 those will be constant necessarily they will be constant in a orthogonal system.

On that line d r will be given as h 1 d u 1 e 1. And the differential arc length dS 1 along u 1 at any point p can be given as h 1 d u 1, this is given as h 1 d u 1. Similarly, if we go along the coordinate axis u 2, we will find that dS 2 is given as h 2 d u 2 and dS 3 along the remaining axis is h 3 d u 3.

Once we have this then the volume element dV can be given as a product of these components. What kind of product gives us a volume element, it is the scalar triple product that we all know, that means, the volume element can be given as h 1 d u 1 e 1 unit vector dot h 2 d u 2 e 2 unit vector cross h 3 d u 3 e three unit vector like this, and because triple product can have a negative sign if we have a particular order and we do not want a negative sign because volume element we want a positive volume element.

So, we take the absolute value of this. And this turns out to be h 1 h 2 h 3 d u 1 d u 2 d u 3. This happens like this, because we know that in orthogonal curvilinear coordinate system, we will always have e 1 dot e 2 cross e 3 equals the absolute value of this will always be equal 1, because they are unit vectors.

(Refer Slide Time: 08:30)



Now, let us consider the gradient, divergence and curl in curvilinear coordinate systems. Suppose phi is the scalar function, and A is a vector function given as A 1 e 1 plus A 2 e 2 plus A 3 e 3, it is a vector it is a vector function of orthogonal curvilinear coordinates u 1, u 2 and u 3. Then we will have for gradient of phi that is written this way is 1 over h 1 del phi del u 1 e 1 unit vector plus 1 over h 2 del phi del u 2 e 2 unit vector plus 1 over h 3 del phi del u 3 e 3 unit vector.

For divergence of the vector function A, we will have this would be given as 1 over h 1 h 2 h 3 del del u 1 of h 2 h 3 A 1 plus del del u 2 of h 3 h 1 A 2 plus del del u 3 of h 1 h 2 A 3. And finally, for curl of the vector function A, it is given as 1 over h 1 h 2 h 3 the determinant of h 1 e 1 unit vector h 2 e 2 unit vector h 3 e 3 unit vector del del u 1 del del u 2 del del u 3 h 1 A 1 h 2 A 2 h 3 A 3 this determinant, this gives the curl.

(Refer Slide Time: 12:18)

Laplacian

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial u_3}\right)\right]$$

Let us also write down the expression for the Laplacian. Laplacian of a scalar function that is del squared of phi can be given as 1 over h 1 h 2 h 3 del del u 1 of h 2 h 3 over h 1 del phi del u 1 plus del del u 2 h 3 h 1 over h 2 del phi del u 2 plus del del u 3 h 1 h 2 over h 3 del phi del u 3 like this.