Physics through Computational Thinking

Professor Dr. Aditya Sharma and
Dr. Ambar Jain
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Lecture 06

Properties of Function

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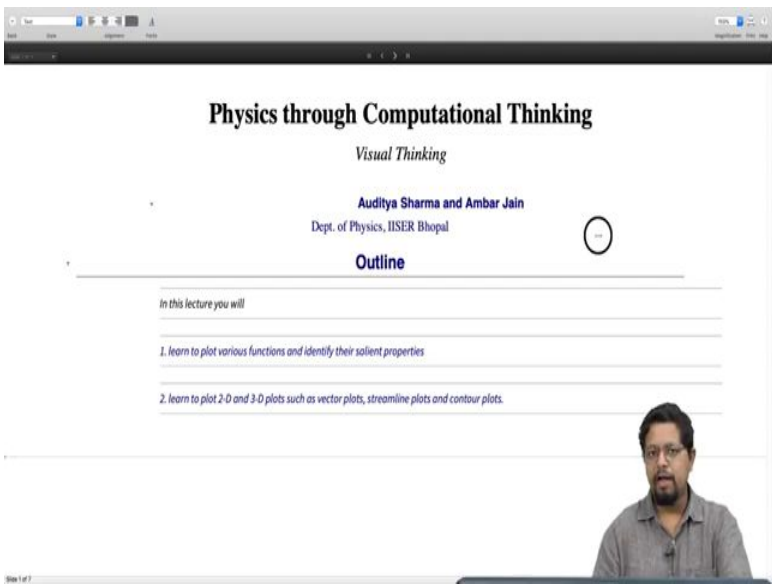

Welcome back to Physics through Computational Thinking. In this video we will continue to learn about plotting and visual thinking. We will learn about salient properties of functions and how to plot them and how to identify them from the plot. We will also learn to do 2 dimensional and 3 dimensional plots for the vector plots, streamlines, and contours. So let us go ahead and get started with this.

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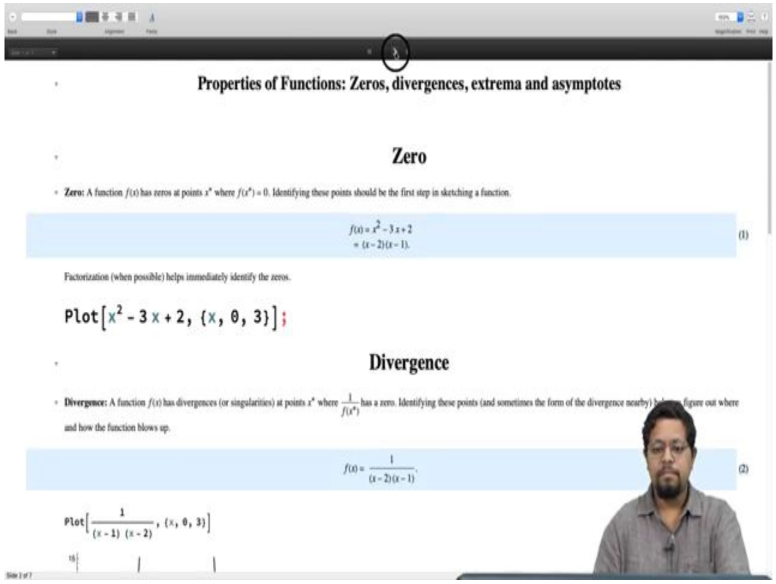

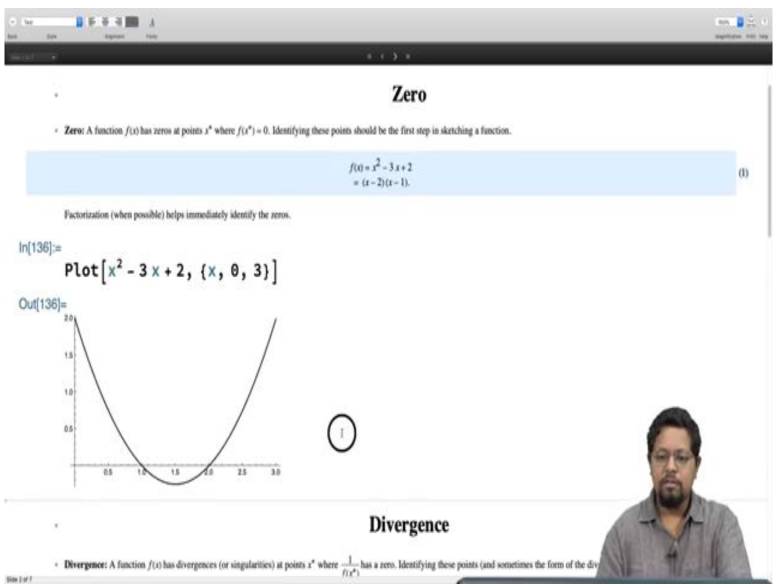

Let us have a quick review of some salient properties of functions such as zeros, divergences, extrema and asymptotes. Zero of a function is defined as a point where f(x) vanishes, if x* is a 0 of f(x) then f(x*) is 0, for example let us consider the function $f(x) = x^2-3x+2$. We factorize this function and we get f(x) = (x-2)(x-1).

From this, you can straight away read, if f(x) is 0 then x* is either 2 or 1. So then x = 2 and x = 1 are 0s of f(x). Let us go ahead and plot it and verify that. You can see that when I make a plot of

x^2-3x+2, this function parabola crosses x axis at 1 and 2 that is where the function vanishes and coincides with the x axis. Zeros are important for many-many things.

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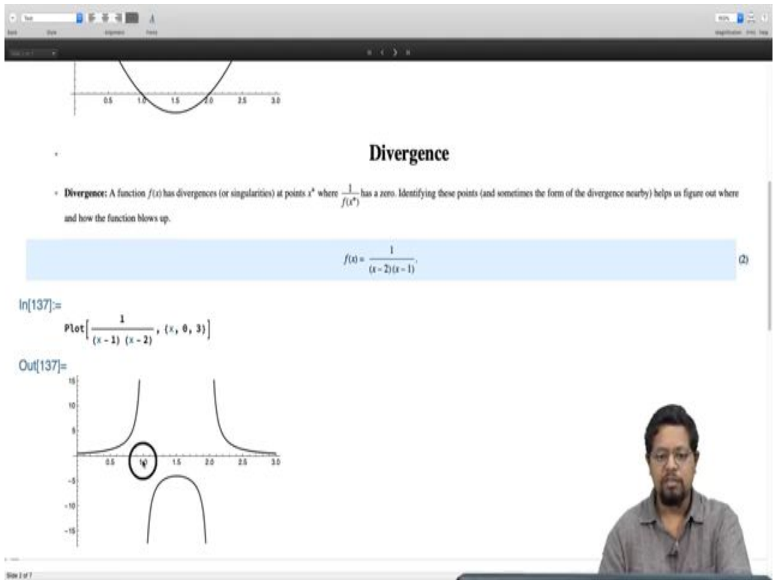

Similarly let us go to the next property divergence. Divergence of a function is defined as a point x*, if x* is a point where this function diverges then 1 over f(x) has a 0 at that point. Let us take for example f(x) = 1/((x-2)(x-1)), f(x) has a divergence if 1/f(x) has a 0, so 1/f(x) is (x-2)(x-1) therefore the divergence for this function happens at x = 2 and x = 1.

Let us plot this function and verify the same, over here you can see that when I plot this function there is a divergence the function diverges where x = 1 and also over here and over here at x = 2 and also from the left hand side. So sometimes the divergences will be positive, sometimes there will be positive and negative depending on which side of the divergent point you are at.

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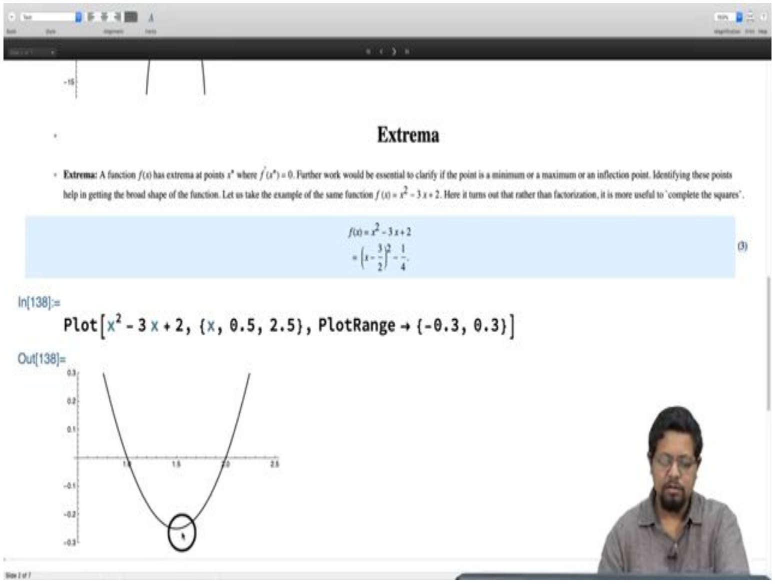

Extremas are another interesting property, extrema is defined as a point x* where f'(x*) vanishes. In order to understand what kind of extrema it is, you need to do some more work, you need to find out the second derivative and from the second derivative you can establish whether it has a minima at that point or a maxima at that point or a point of inflection.

In this case, let us take the same function again $f(x) = x^2-3x+2$. For this simple example, let us understand where is the maxima or minima for this function. At this point it is actually useful to complete the squares and then factorize f(x). When you complete the squares for this function you see that f(x) becomes $(x-3/2)^2-1/4$. Now this function becomes minimum when x goes to 3/2.

So let us go and plot it and verify this, so when I plot this function, again you can see it has got a minima at x = 3/2.

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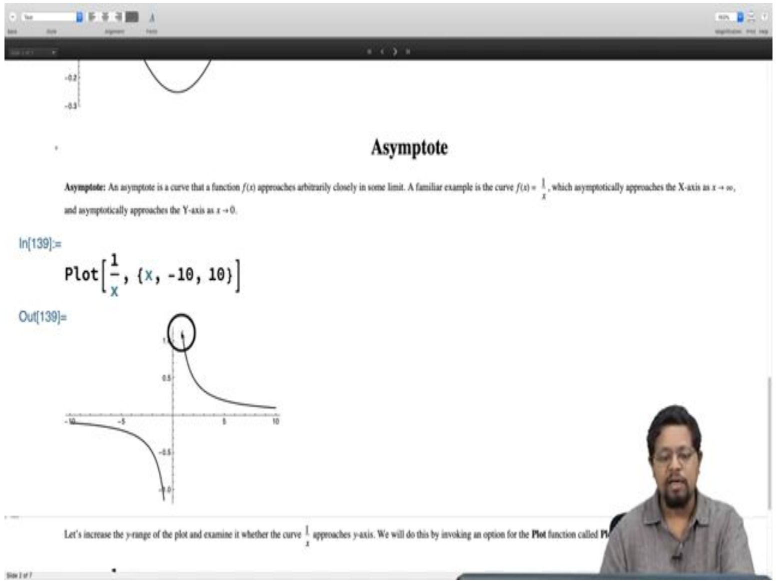

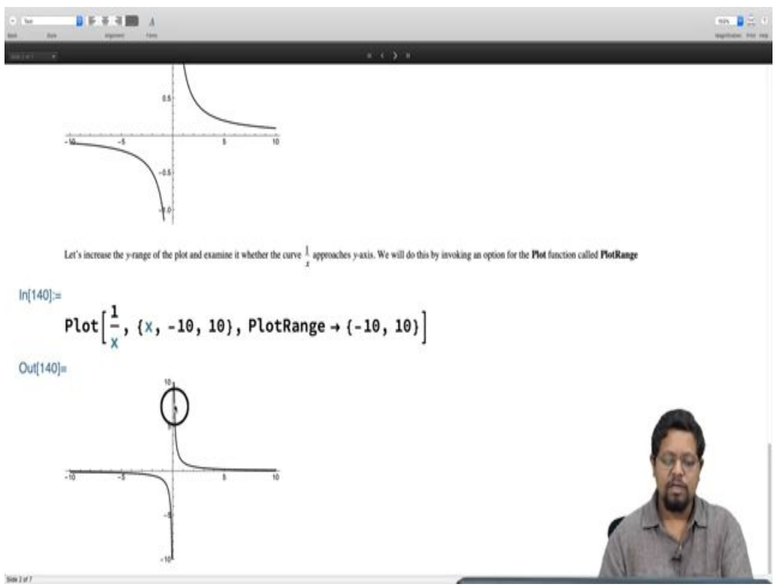

Similarly asymptote is also a property of function. An asymptote is a curve that a function f(x) approaches arbitrarily close in some limit. For example x goes into infinity, x goes to minus infinity or x goes to zero, what happens to f(x) in that limit that is called asymptote.

Asymptote is a line which f(x) approaches when x becomes arbitrarily close to some limit. A familiar example is f(x) = 1/x which asymptotically approaches x axis as x goes to infinity and asymptotically approaches y axis as x goes to 0. Let us plot this and find out, so the plot of 1/x as x goes to plus infinity, we see that the function f(x) becomes arbitrarily close to the x axis in fact in the limit x goes to infinity it becomes parallel to x axis.

Similarly as x goes to 0 the function becomes arbitrarily close to y axis and in the limit x goes to 0 it becomes parallel to the y axis, on the left hand side x goes to minus infinity the function becomes parallel to the x axis and x goes to 0 it becomes parallel to the y axis, it approaches y axis. So x axis and y axis are are called asymptotes of 1/x, x axis is a asymptote of 1/x in the limit x goes to plus infinity or minus infinity and it is a asymptote, y axis is a asymptote of f(x) of 1/x as x goes to 0.

We can further increase the range of this plot and see that they actually do become arbitrarily close but they never actually meet. So the function f(x) keeps on getting close and close to y axis becomes arbitrarily close but it never really intersects with it. So these were some of the salient properties of the function we are going to take, we are going to study zeros, divergences, extrema and asymptote using some examples.

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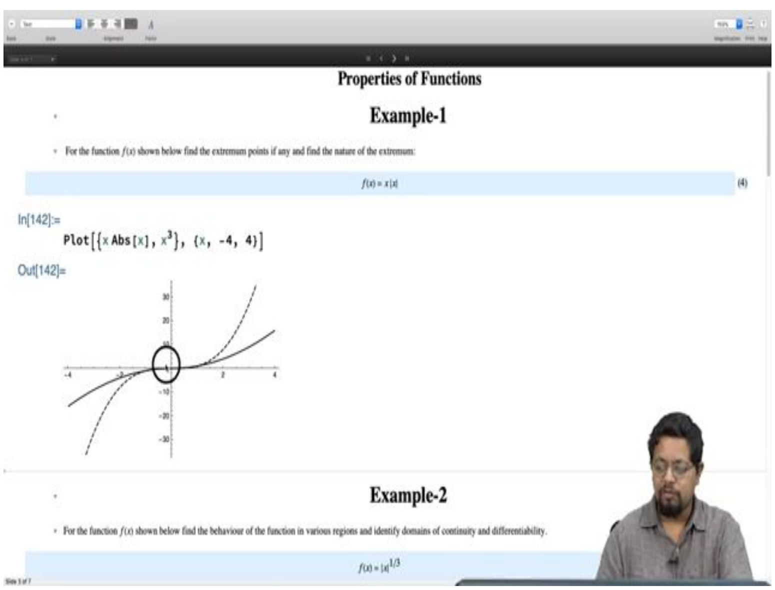

Let us take the first example and understand properties of functions: zeros, asymptotes, divergences etc using these examples. First example is f(x) = x |x|. Can you find out what are the salient properties of this function, let us go ahead and plot this function and find out what are the salient properties of this function, where are the zeros, extremas, maximas, minimas, asymptotes etc.

So when I plot this function for |x| you can use absolute value of x, so the function I am plotting x |x|, absolute value is represented by Abs and I am plotting the range x, -4 to 4. So this function is x^2 for positive x axis and $-x^2$ for negative x axis, so the question is where does it have a 0? Very clearly, it has only one 0 and lies at x = 0.

Where does it have, the question is: is that a 0, a maxima, minima or a point of inflection? That will depend on what happens to the second derivative? In this case we have got function to be flat over here in fact its slope is 0 at x equal to 0 and also the second derivative will turn out to be 0, so this will become a point of inflection. You can also plot x^3 and you will see interesting enough that this has got similar kind of behavior as x^3. x^3 is also a function which has got a point of inflection at x = 0.

Let us take another example, also lets quickly check other properties. It does not have any asymptotes, the function diverges as x becomes very large or very small, it does not have any other divergence apart from divergence at x equal to plus infinity or minus infinity it has only one 0 and that 0 is also a point of inflection.

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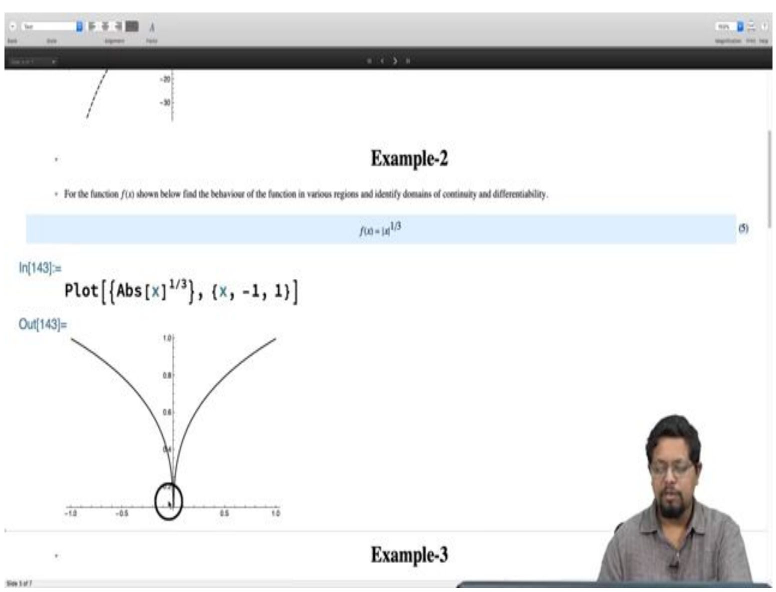

Let us take the second example, let us take the function $|x|^{1/3}$, and let us find out what is the domain of differentiability and continuity for this function, is this function continuous and differentiable everywhere. To do that let us go ahead and plot this function. When we plot this function we see that it forms a cusp at x = 0. So first of all is it continuous?

It is clearly continuous on the left hand side, it is clearly continuous on the right hand side but is it continuous at this point that depends on if if the function have the same limit as you approach it from left or as you approach it from right, absolute value of x vanishes when you approach it from left or approach it from right. Therefore absolute value to the power 1/3 also vanishes, so so both the functions approach 0 from the left side and right side from both directions, the function evaluates to 0.

Therefore it is continuous at x = 0 but clearly it is not differentiable at x = 0 because the derivative over here has a large negative value and derivative over here tends to the large positive value. Therefore the function is not differentiable at x = 0, it is differentiable for $x < 0$ and it is differentiable for $x > 0$ but it is not differentiable at x = 0.

In this case the zero is also at x = 0 but it is not a extrema point because f' is not defined you cannot say f'(x) = 0 but it is a absolute minimum value for the function at this point, it does not have any divergence apart from x equal to plus infinity and minus infinity.

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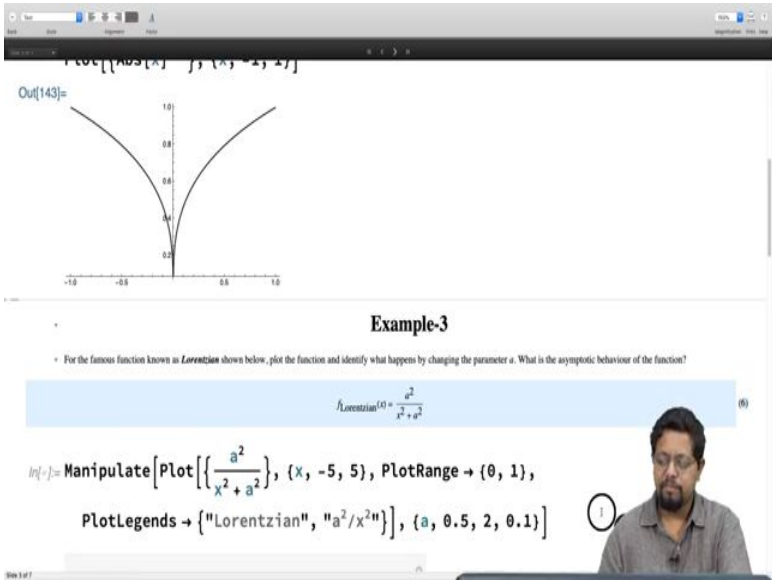

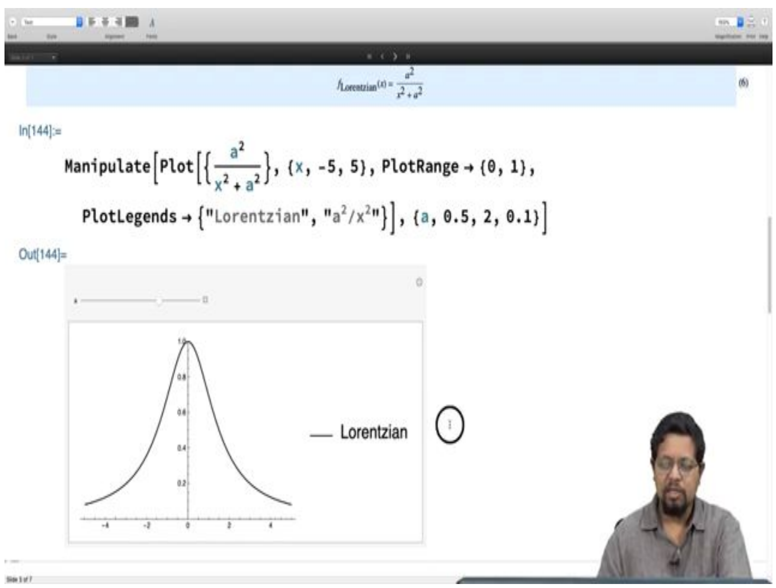

Let us take the next example, this time we will work with the famous function called Lorentzian. Lorentzian is given by $f(x) = a^2/(x^2+a^2)$. Let us find out what is the asymptotic behavior of this function and at the same time also look at what are its other properties. So in order to do that what we will do is, since it is parameter 'a' we will also understand what the parameter 'a' does using the Manipulate construct.

For that I am going to use the Manipulate command, I will manipulate the parameter 'a' in the range of 0.5 to 2 and I will plot the function $a^2/(x^2+a^2)$ in the range of -5 to 5. When I execute that, I get this frame and I slide ‘a’ and see what happens, the parameter ‘a’ determines the width of this curve, Lorentzian is a curve which vanishes at plus infinity and minus infinity.

As you increase ‘a’, its width increases. So the parameter ‘a’ determines the width of the Lorentzian, it has a maxima at x = 0 you can easily verify that the derivative of this at x = 0 vanishes and also the second derivative is negative and therefore it has got a maxima at x = 0. As x goes to plus infinity or as x goes to minus infinity, the function gets arbitrarily close to the x axis therefore it is asymptote is the x axis.

Now Lorentzian is an interesting function. Let us go ahead and look at another example and we will come back to Lorentzian in a moment.

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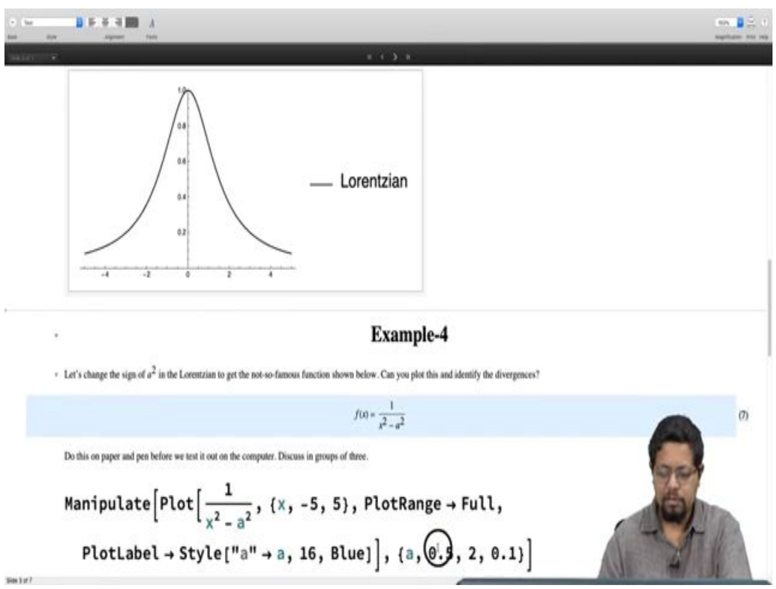

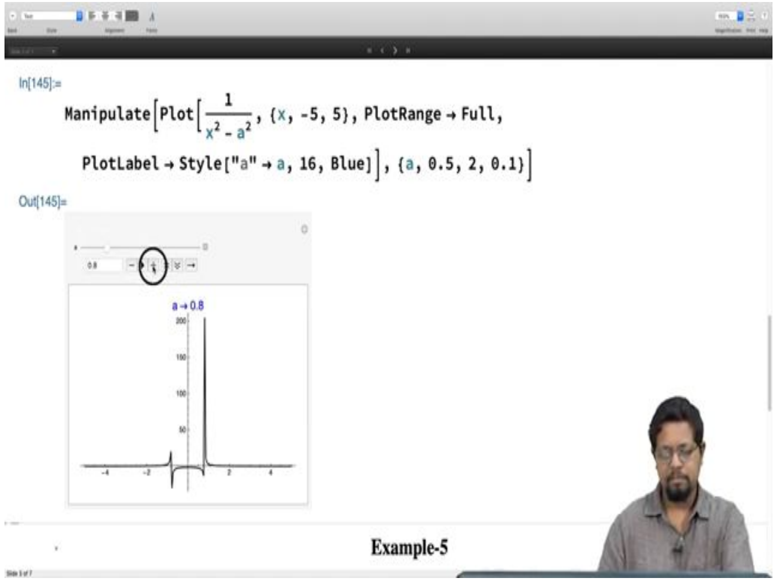

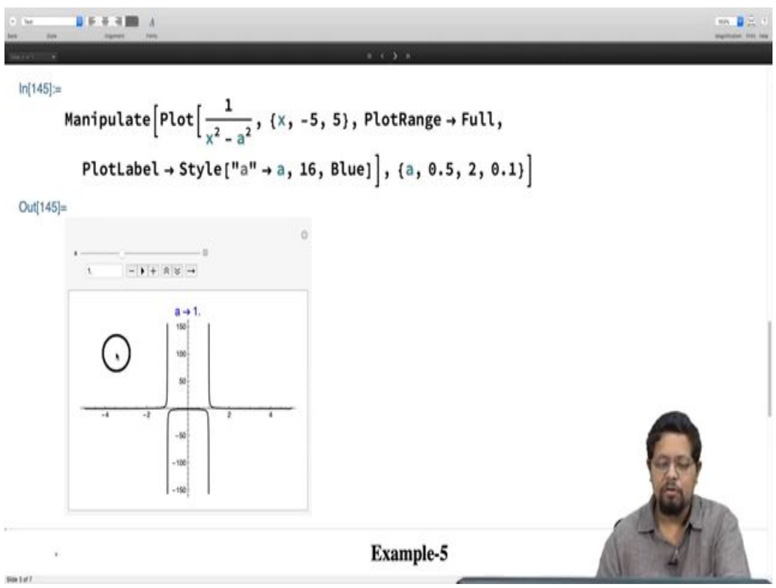

Let us look at the example where we take a function just like Lorenztian but in the denominator will keep it x^2-a^2 than x^2+a^2. When it is x^2+a^2, it is a Lorentzian function, when it is x^2-a^2 this function has divergence let us identify where the divergence is. We will go ahead and plot this function and you can go ahead and slide the slider ‘a’.

As I slide ‘a’, I will do it in steps because it takes a moment to plot. As I increase ‘a’ you see the point at which the divergence is happening also increases. In fact the divergence is exactly equal to x = +a and -a. So this is an example of the function which has got a divergence, it has also asymptotes for example as ‘a’ gets arbitrarily close to +1 in this particular case as x goes arbitrarily close to 'a', in this case 'a' is 1, as x goes arbitrarily close to 'a', this approaches the line x = a which is parallel to the y axis.

Similarly when x becomes arbitrarily close to a = -1 the function approaches the line x = -1 and gets arbitrarily close to it. So x = +1 and x = -1 are asymptotes for this function and also x = 0, sorry the x axis that is y = 0 is also an asymptote in the limit x goes to plus infinity and minus infinity. So this particular example, it has got 3 asymptotes, y = 0, x = +a and x = -a.

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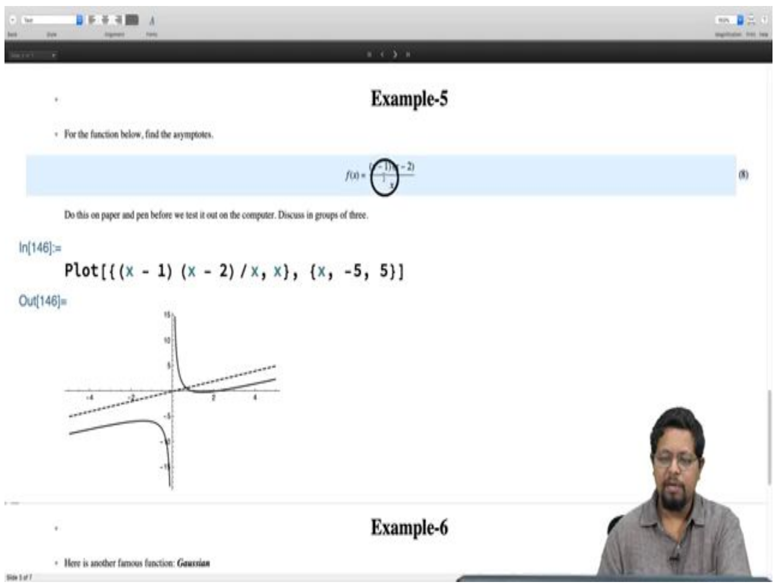

Let us go to the next example, let us take the function f(x) = (x-1)(x-2)/x and find its asymptotes. Let us go ahead and plot this function in the range -5 to 5. In order to find the asymptotes we have to take a limit, so for example for this function if I take the limit x goes to 0 you see the denominator dominates the numerator which goes to a constant which is +2 so the function becomes 2/x, it behaves like 1/x in the small x limit.

So it's asymptote is y axis in the small x limit. In the large x limit this 1 and this 2 can be ignored and I get a x^2/x is approximated as x, so in the large x limit it is asymptote is x equal to y line which is represented by this dash curve. So we can find asymptotes by taking various limits, so plotting helps in visualizing and understanding what kind of limits I should take.

Without plotting you can still do this exercise but it can be slightly harder, so with visualization plotting it becomes easier to understand various properties of the functions. Let us go ahead and take a look at another example.

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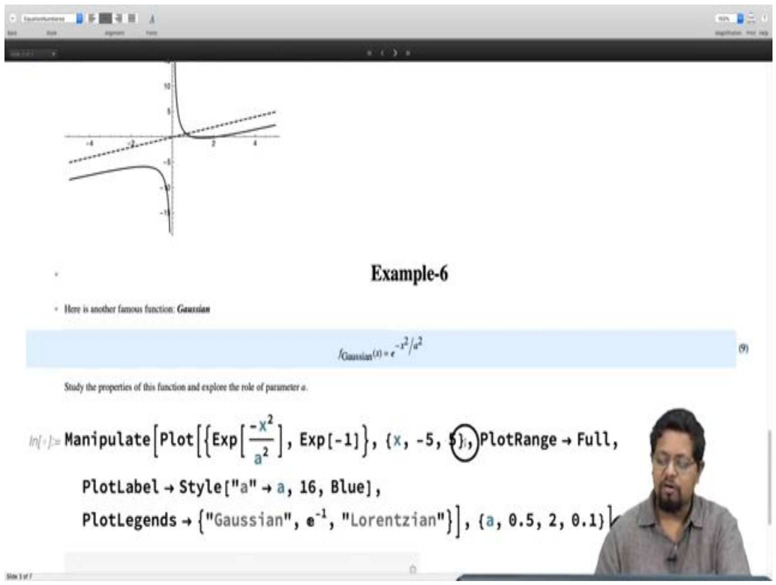

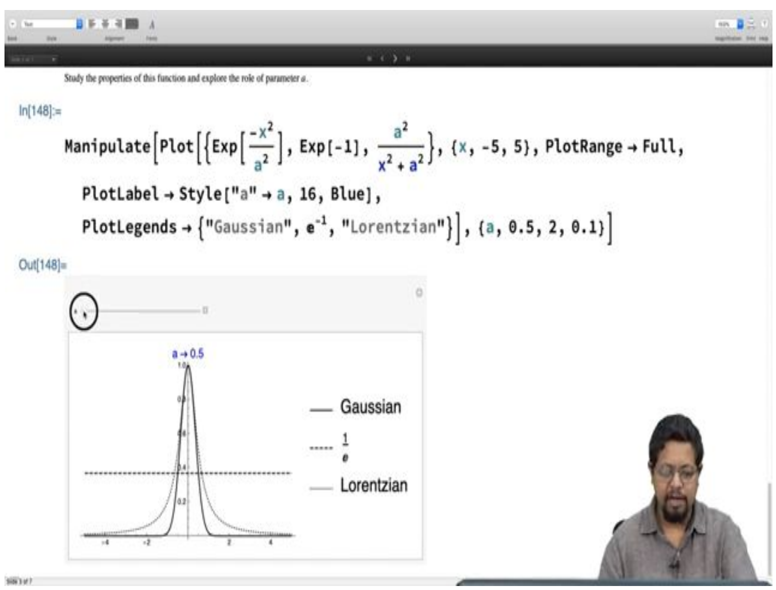

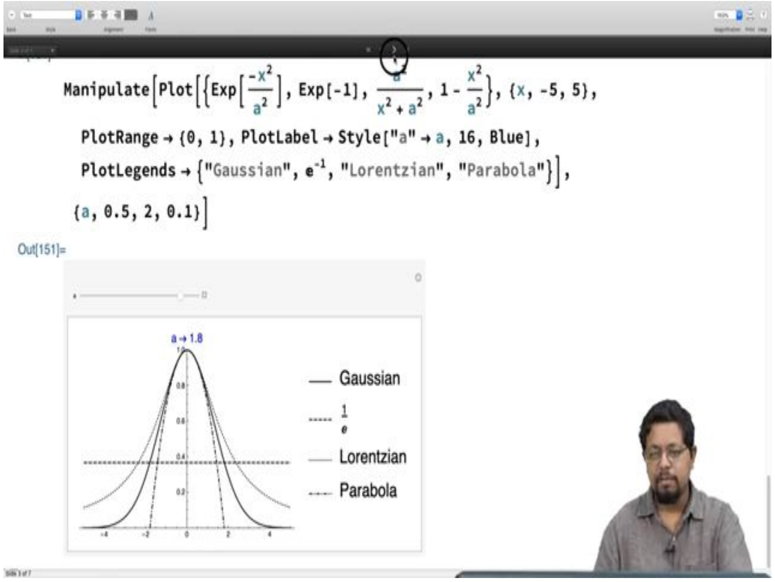

Again this time we will take a famous function called Gaussian. You may be familiar with this function. Gaussian is given by an exponential of $-x^2/a^2$. Let us understand the salient properties of this function, so let us go ahead and plot it, this time with a Manipulate, and it will manipulate the parameter 'a' in the range of 0.5 to 2. Let us go ahead and plot it, as I increase 'a' the width of the Gaussian increases, I have also plotted 1/e line, this dash line corresponds to 1/e, this corresponds to the value of 'a' by which time x^2/a^2 has become 1.

So as x starts from 0, outwards towards positive x or negative x, this point represents when x^2/a^2 has reached 1, that is when the value of the function is dropped from its maximum to 1/e and that is a point which gives the width of the Gaussian and this point corresponds to x = a. Let us go and compare this because this is a falling off function let us go ahead and compare this with Lorentzian that we just plotted.

So add Lorentzian to this, Lorentzian was $a^2/(x^2+a^2)$ and let us go ahead and change 'a', you can do it yourself and play around with it. You can always see what the value of 'a' is. Gaussian is falling up much faster compared to the Lorentzian, in fact in the large x limit Lorentzian behaves like $1/x^2$ while Gaussian falls off exponentially $e^{-x^{\wedge}2}$ which is much faster than $1/x^2$.

So Gaussian will always go to 0 faster than Lorentzian that is the large x behavior but for both of the functions the maxima appears at x = 0 and both the functions have asymptotes in the x goes to plus infinity or minus infinity as the x axis. Also in the neighborhood of x = 0 both the functions Lorentzian and Gaussian behave like an inverted parabola.

In order to see that let go ahead and plot $1-x^2/a^2$ and we have to constrain the plot range, we can say the plot range is 0 to 1 there we go and as I change ‘a’ you see the behaviour of all the 3 functions: the parabola, the lorentzian and the Gaussian is identical near x = 0, as only as you go away from x = 0 you see the function behavior is changing.

I can also add a parabola, now my figure is complete I can change ‘a’ and see that their behavior always is identical near x = 0 the 3 functions all of them behave quadratically near x = 0 and their behavior changes as x goes away from 0, 0 the parabola diverges, Gaussian converges to 0 the fastest, Lorentzian also converges to 0 but slower than the Gaussian. Those were some of the examples where we studied the salient properties of the functions, that will be all for today and we will continue with more interesting things next time.