Physics through Computation Thinking

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Lecture 47

Random Walks 4

Hello everybody. So we have been discussing random walks and so we saw how some of these key results namely diffusive motion how, if you take n steps on average, you only cover of order of $\sqrt{N}$ distance. This is one of the characteristic features of diffusive motion and it comes about it with just simple random walk considerations, we were looking at 1 D and we so saw some exact results. We also saw some ways of simulating this and then comparing one against the other.

So, what I want to do is, use this so called Stirling approximation, which we mentioned in passing and you know to see how these, some of these analytical results which come out of the Stirling approximation are absolute excellent approximations and check against the numerics and see that in fact they work out very well. . So, I want to explicitly do this, so we suggested this as an exercise in the past but so today in this module we will go into some of the details of this.

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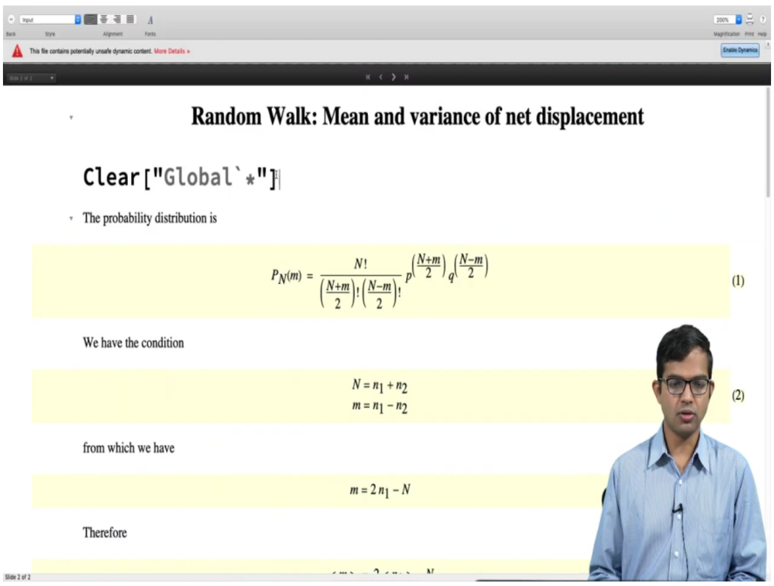

$$P_N(m) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\, p^{\left(\frac{N+m}{2}\right)} q^{\left(\frac{N-m}{2}\right)} \quad (1)$$

We have the condition

$$N = n_1 + n_2$$
$$m = n_1 - n_2 \quad (2)$$

from which we have

$$m = 2n_1 - N \quad (3)$$

Therefore

$$\langle m \rangle = 2\langle n_1 \rangle - N = 2N\left(p - \frac{1}{2}\right) \quad (4)$$

Also the variance is now given by

$$\langle m^2 \rangle - \langle m \rangle^2 = \langle (2n_1 - N)^2 \rangle - \langle 2n_1 - N \rangle^2$$
$$= \left(4\langle n_1^2 \rangle - 4N\langle n_1 \rangle + N^2\right) - \left(4\langle n_1 \rangle^2 - 4N\langle n_1 \rangle + N^2\right)$$

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$$= 4\left(\langle n_1^2 \rangle - \langle n_1 \rangle^2\right) \quad (5)$$

Therefore

$$\langle m^2 \rangle - \langle m \rangle^2 = 4Npq$$

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Therefore

$$\langle m^2 \rangle - \langle m \rangle^2 = 4Npq \quad (6)$$

Special Case: The unbiased random walk

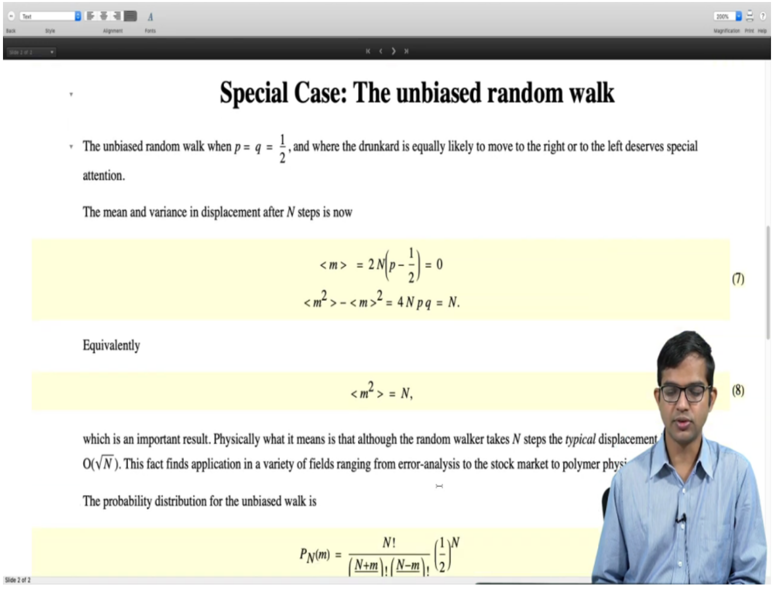
- The unbiased random walk when $p = q = \frac{1}{2}$, and where the drunkard is equally likely to move to the right or to the left deserves special attention.

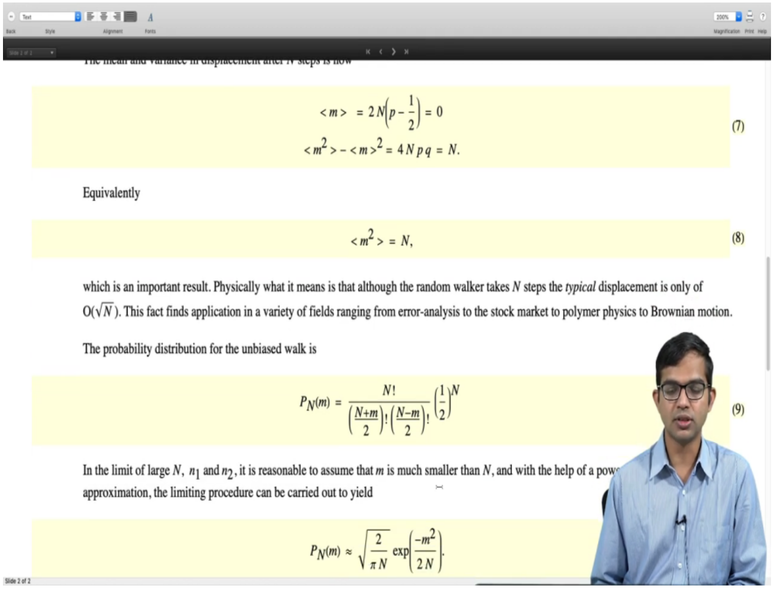
The mean and variance in displacement after N steps is now

$$\langle m \rangle = 2N\left(p - \frac{1}{2}\right) = 0$$
$$\langle m^2 \rangle - \langle m \rangle^2 = 4Npq = N. \quad (7)$$

Equivalently

$$\langle m^2 \rangle = N,$$





Ok, so as always, we start by clearing mathamatica of its past memory. So, a quick recall of you know how random walk happens and how you can extract mean and variance of net displacement. So, we have, we imagine a walker who starts at the origin and with probability p he moves to the right and with probability q = 1 - p he moves to the left after every time step.

So, there it is discrete time, discrete space, so he covers 1 unit of distance in each move and at time of 1-unit time elapses before he makes another move and there is no option for, for the random walker to not make a move. He has to either go to the right or to the left and then we looked at how you can study the quantity like n1 and n2 and define n1 as the number of steps which are taken to the right and then crucial quantity becomes m = n1 - n2 which we can solve for m in terms of n1 and N.

And then we saw that the average of m is simply given by 2N(p - 1/2). And then we also saw that there is a nice connection between the variance of this random variable m and the variance of the random variable n1. So infact one of them is just 4 times the other, like here. And so the general expression it turns out is, $\langle m^2 \rangle - \langle m \rangle^2 = 4Npq$.

And then we quickly looked at the unbiased case, when you have p = q = 1/2 and then basically, since average of m is anyway going to be 0 for this case, p = 1/2, then you get the very important result for random walks that $\langle m^2 \rangle = N$. It is just exactly N. So, this is what we saw and then we saw that you know this has applications in many fields and sub-fields of physics and allied areas.

And then we also said that it is possible to do better, not just make a statement about the average or the average of the square of the random variable. In fact you can get the whole probability distribution for the unbiased random walk and that turns out to be this binomial distribution. And then we said, we claim that if you do use the Stirling approximation you can, you can get it to take this gaussian form.

So one question which would arise is, of course the precise method one uses to go from equation 9 to equation 10, which is what I gave you as homework, but another is to ask how reasonable is 10 with respect to 9, ok, maybe it is an approximation, how good an approximation is it? So this is the agenda for this module, to explore this question.

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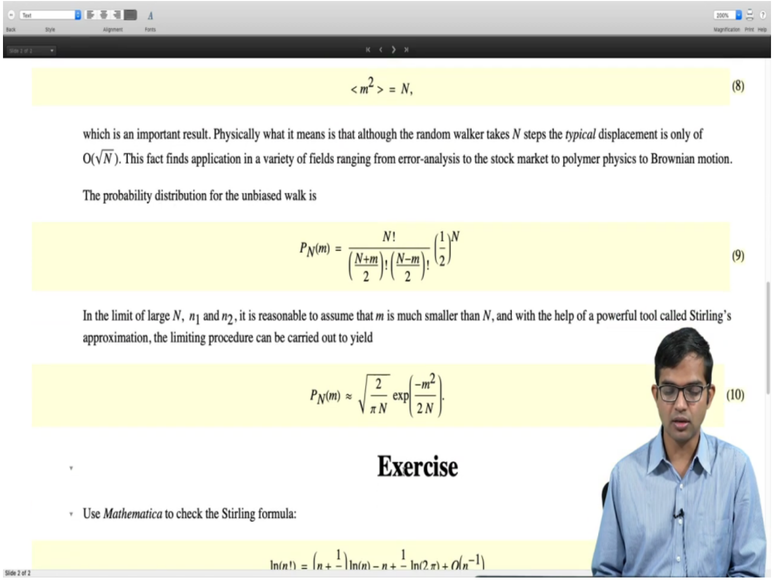

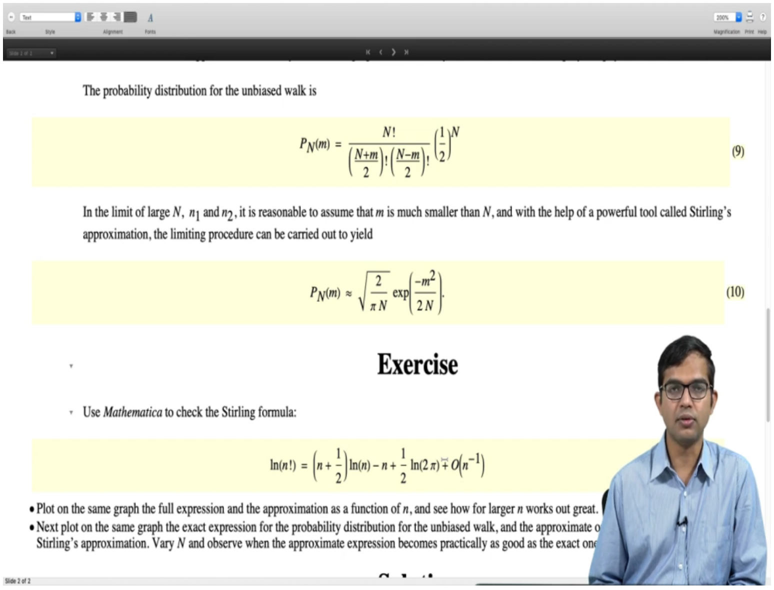

And, infact it turns out that you can do this check, I had asked you to do this analytically by hand, but you can even ask mathamatica to do this for you. Right, I mean, so that is also something for you to play with, you can actually try to use some commands like simplify and log of and factorial. Not something that I am going to do now, but you can try it and see you can get Mathematica to suggest for you approximate forms.

Ok, so what I want to do here is actually just use the plot command. So, infact if you want to pause the video here and carry out these two tasks which I have laid out for myself and it will be good if you can do it your own way and then cross check against mine. Maybe you have better way than what I have.

So, plot on the same graph the full expression and the approximation as a function of N and see how for larger N this works out, that's the first one. So, I have plot exact, I am going to call it plot exact and then I am going to make a list plot of this table. So, table is a, so N as you know, if I have to define N!, it must, N must be an integer.

So, if I stick to integer values of N, then factorial is of course very well defined, but it turns out that there is a generalisation of this. You can actually play this with mathamatica and check for yourself that, infact mathamatica is happy to compute factorials of non-integers for you. So, there is a generalization of the notion of a factorial. If you have not already seen it, you might you know check this out in the maths method type of text book.

Alternatively, even better is for you is to play with this. Just you know use the factorial function and plot it on mathamatica, plot just the factorial as a function of N and not ListPlot like I have here. You can even use just the regular plot. But here for simplicity I am choosing factorial of n and n going from 1 to 10 in a discrete manner and I create a table of this. Table will generate for me an array, and then I will use ListPlot to plot all these, you know the discrete numbers, right so that's what plot exactly does. And then I can go ahead and compare against plot approx.

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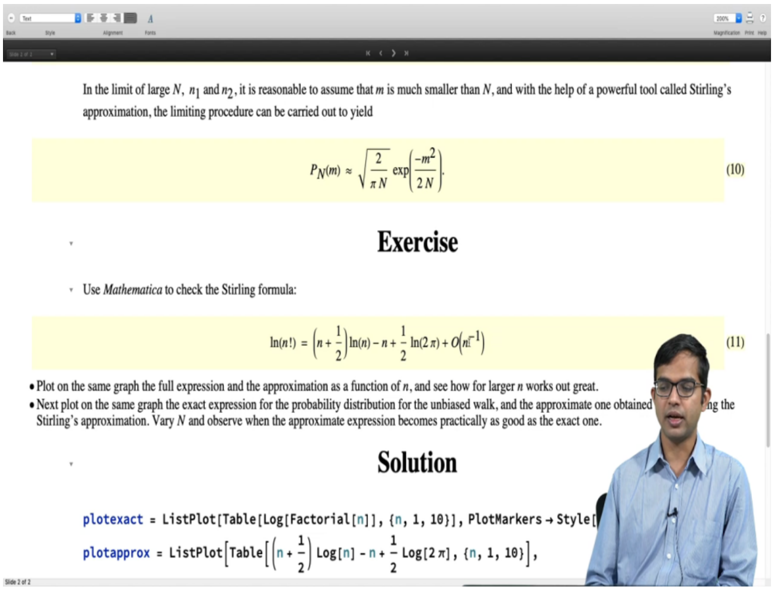

As you can see the right-hand side of equation 11 does not give you any undue restrictions on what N should be. Even if you, let us say you started out with no knowledge of you know the generalization of the factorial function, so the left hand side seems to demand that N should be an integer but the right hand side there is no such restriction as you can see.

And so infact there is a way to generalise your N!, the idea of N! such that this formula would work out in any case for arbitrary real values of N. OK. So, I have here a list plot for the exact function after it has been discretised and put it into an array using the Table command. So, like wise I have a table of the approximate function of the right-hand side. So, now the question is, how large must N be, before these two functions start to agree with each other.

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So let me run this, and then I will use the Show command to make both of them appear on the same same graph. And, as you can see the agreement infact becomes extremely good even for very very tiny values of N. It is not like n has to be some huge number before you start seeing it, you can see that even for n equal to 3 or n equal to 4 already this is an excellent approximation. And, so indeed for large n basically these two are indistinguishable for all practical purposes they will be the same really.

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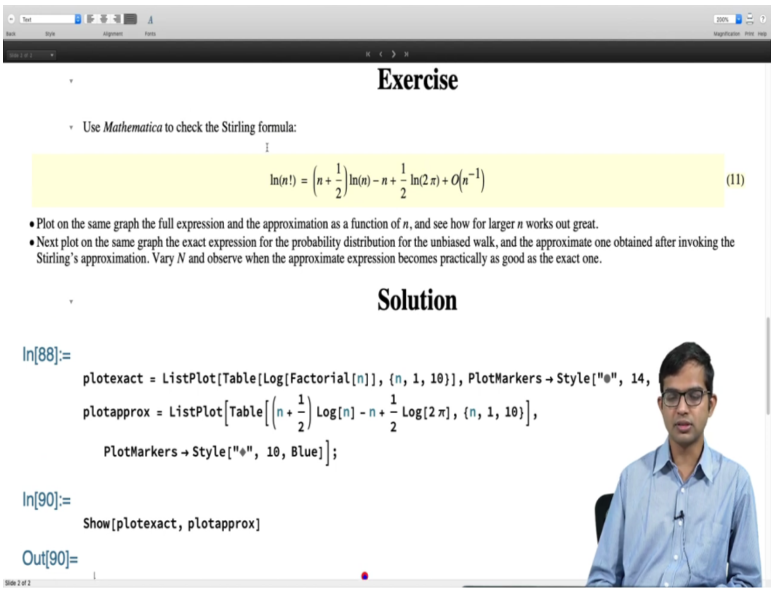

So sometimes you might encounter the Stirling formula in a slightly different way, so this has some more information in it which actually, you can actually neglect you know one of these terms for very large values of N. So you can infact take an exponential of this whole thing on the left hand side and on the right hand side, then you can write it as you know $N! \approx N^N \epsilon^{-N}$.

So, time is another factor which can be neglected, so that is also something for you to check. In fact what you should do is, make a plot of N! like I said earlier and check first of all that this factorial is a reasonable function for arbitrary N and not just for integer values of N, that's the first things to do.

Second is to plot on the same graph $N^N \epsilon^{-N}$. So, if you can plot this function which clearly also can take any real value of N. There is no restriction that n should be integer. Both of these and check for yourself that indeed this, these two curves will agree with each other very well, that's going to be your exercise problem.

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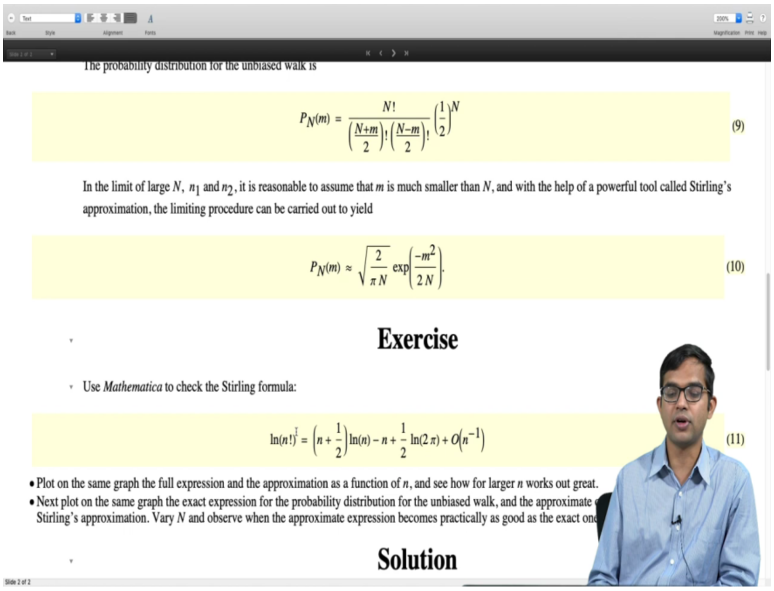

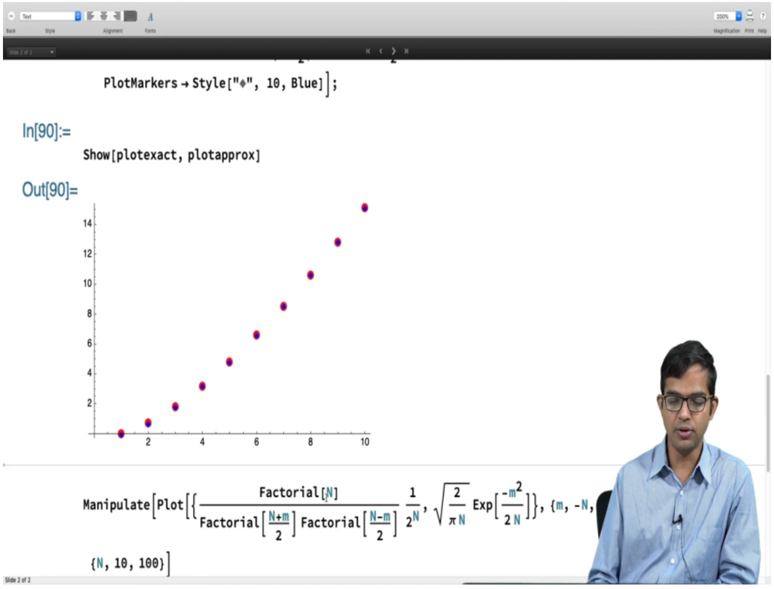

Ok then next what I want to do is compare, you know we have used, I am claiming that if I use this, this is also something which I will allow you to verify for yourself. If I use the Stirling approximation, as in equation 11 and plug it into equation 9, there is a way to go to equation 10, that's for you to verify. And, but here what I want to do is again use the plot function, to plot equation 9 and equation 10 and then see how well they super impose upon each other and in what limit, what happens.

That's what I want to do, that's the second exercise. And in order to do this I will use the Manipulate command because I want to vary my N itself, so I will allow my N to go from 10 to a 100 and then I will plot each of these distributions, one of them the approximate and other one is the Gaussian form and take N to go from -N to +N. So, I will generate this and then I see N appears in many contexts, so let me use this again and then it should be fine. Yeah, so it is fine.

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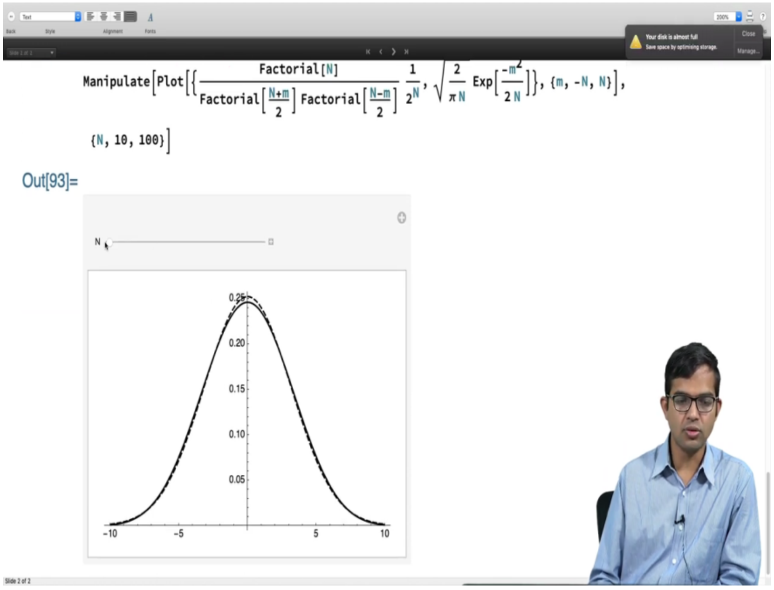

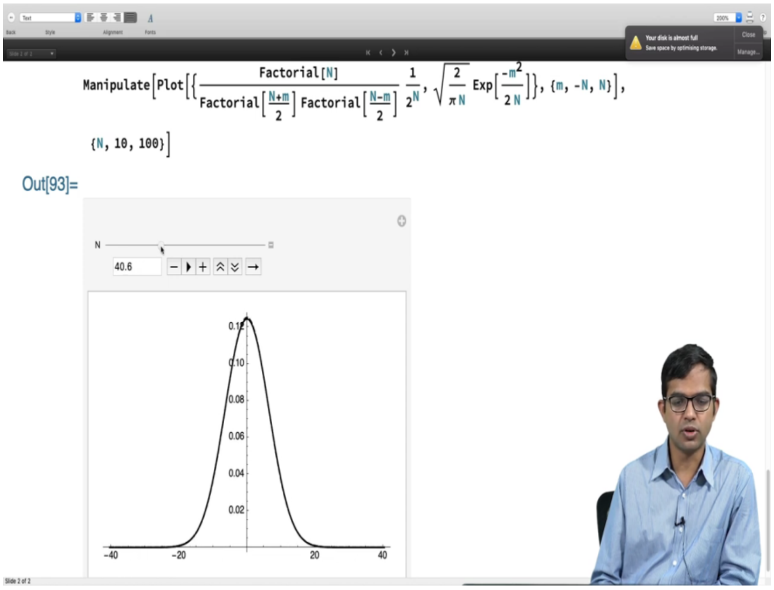

So now I can go ahead and vary N. So there you see, after a point the two curves are just totally indistinguishable. So you see that at this value of N, whose value you can check if you, so if I put N equal to around 15, I can see that there is a some some distinction between the two, when N = 10 for sure you can see, but when n equal to may be around 30 or 40 let us say, already the two are basically indistinguishable.

For N = 50 and so on, it becomes better and better and you do not even realise that there are two curves there, it is just one, they all look the same. Ok, so this is something for you to play with and check for yourself that indeed this distribution function obtained.

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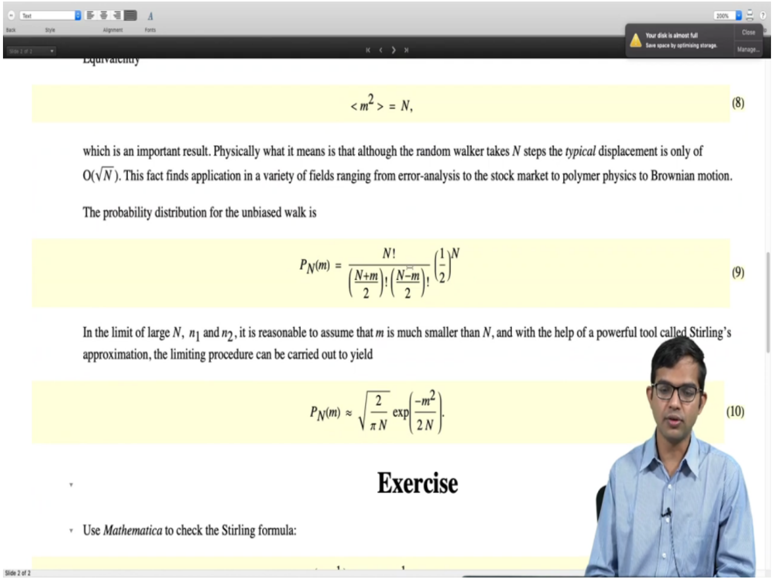

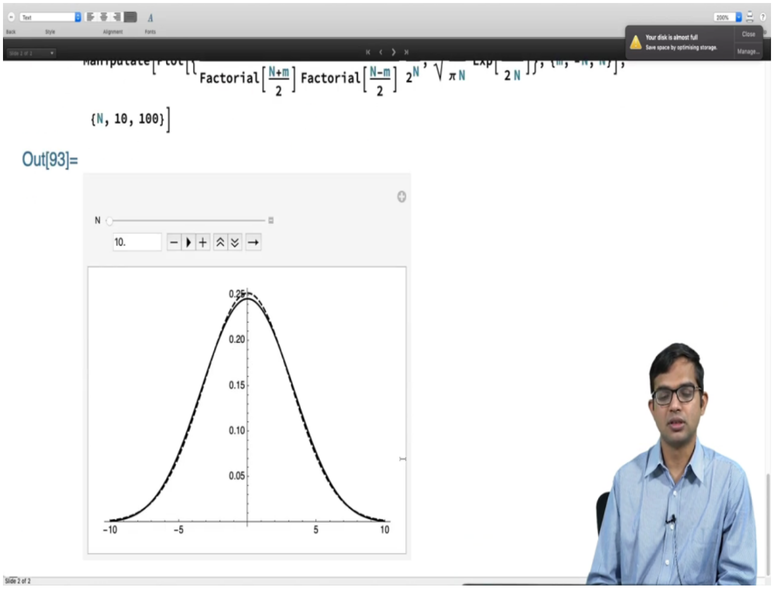

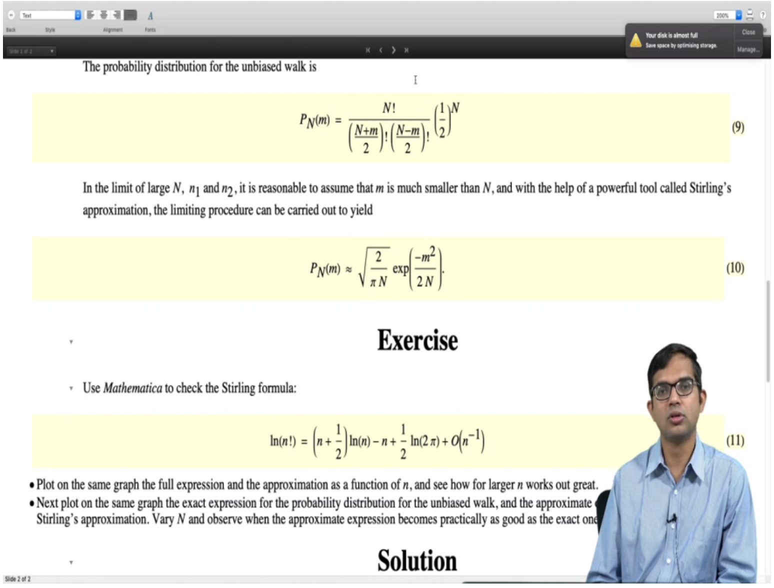

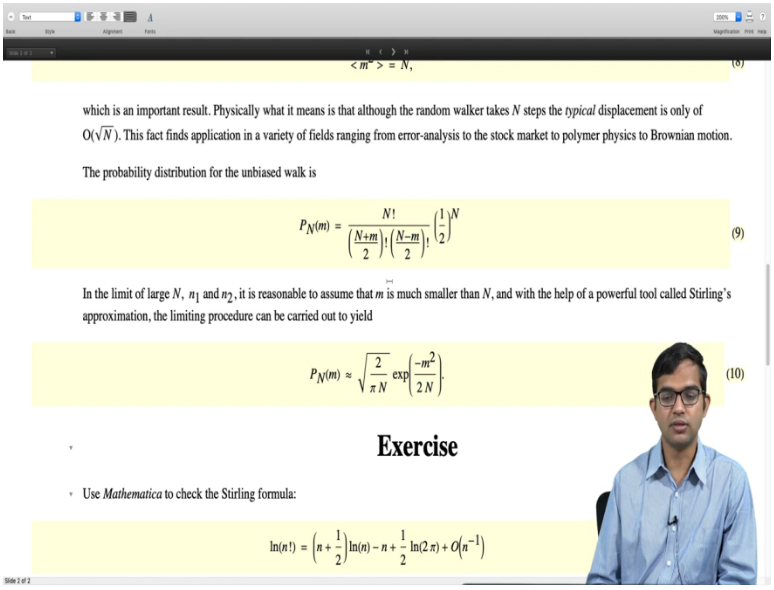

Here of course, you know m will take only discrete values, but on the other hand here it takes all values. So the point is that when N becomes large, you know the difference between adjacent discrete numbers becomes so tiny so, and in that sense this becomes, your distribution will also become a continuous distribution. It is not just for the discrete values of N or discrete values of m, but you can treat m to be a continuous degree of freedom.

Alright, so this was just a quick module, where we revisited some things which we already saw and then I am giving you a sort of a motivational direction. So, to speak to check these things for yourself. But there are still a few more things which I have left undone or which remains to be done and I will be leaving that for you guys. One of them is to actually verify that equation 10 comes from equation 9 if you use the Stirling approximation.

And the second is to test for the equivalence between you know these two forms N! and $N^N e^{-N}$ using you know similar plotting and trying to super impose one on top of the other. Alright, so that was a short module on this kind of comparison. Thank you.