Physics through Computational Thinking
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Lecture 04
Functions Behavior near Extrema

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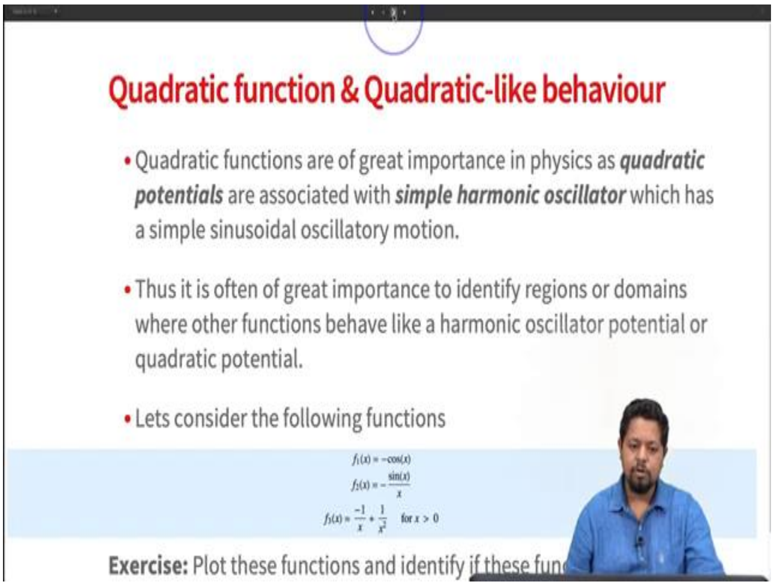

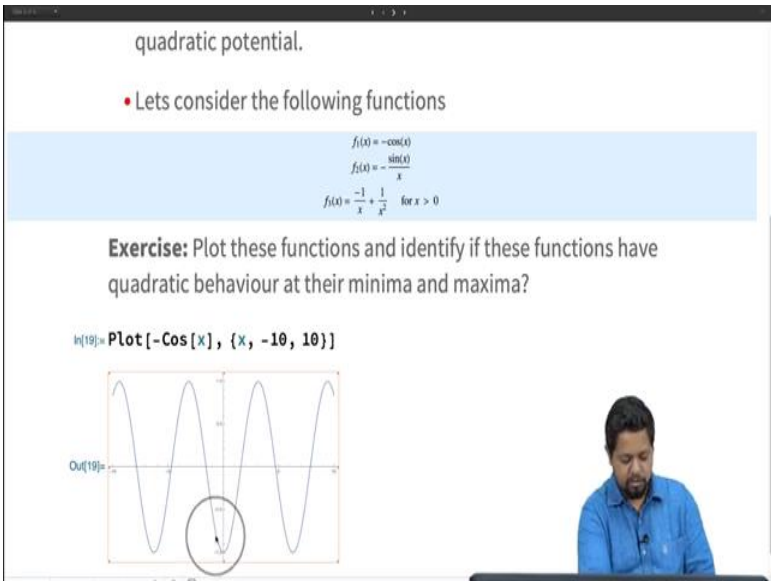

Let us go ahead and find out what other functions have quadratic-like behavior or what are the functions which you can call quadratic or if there are functions in the vicinity of their maxima's and minima's, we have a quadratic-like behavior.

Quadratic-like behavior is something that is most commonly expected around minima or maxima of a function. So, in this little exercise, I have given you 3 functions. What we want to do is to plot these 3 functions and identify whether these functions have quadratic behavior at the minima and maxima.

Alright so, what is the first thing we are going to do? We are going to plot the function. So, let me go ahead and plot the first function which is -Cos(x). Let us plot it from x goes from -10 to 10. We get an oscillatory function which is -1 at x=0. And then it peaks up again over here and then peaks at п and then at 2п, it has got its minima again at a period of п.

So, at 0 it has a minima and 2п it is a minima and at п it is a maxima. So, we want to find out what is the behavior of minima at this point. What we will do is, we will first go ahead and solve it analytically.

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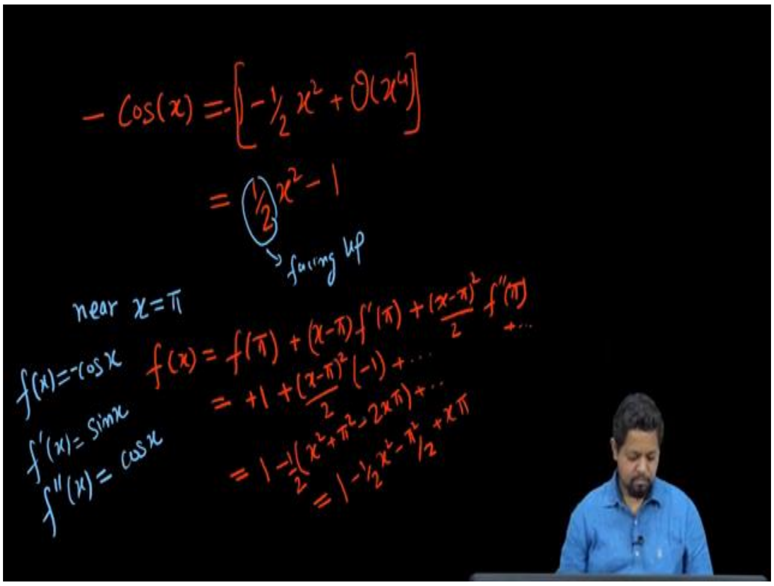

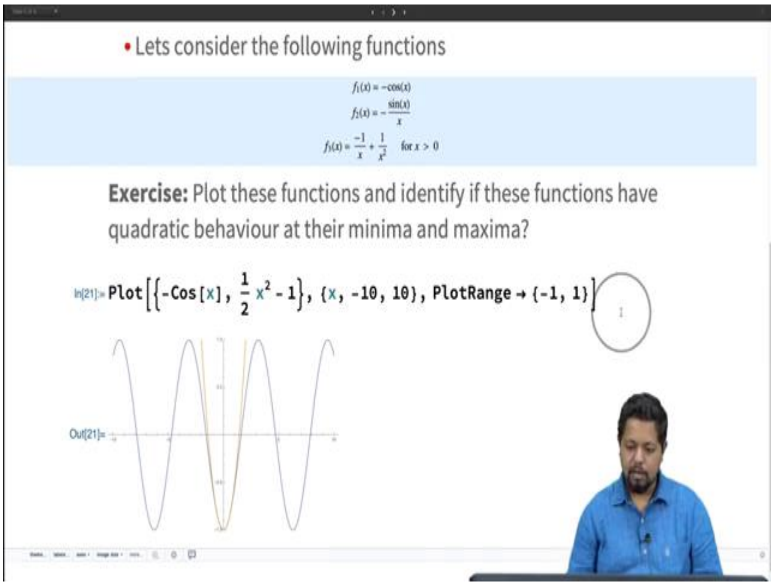

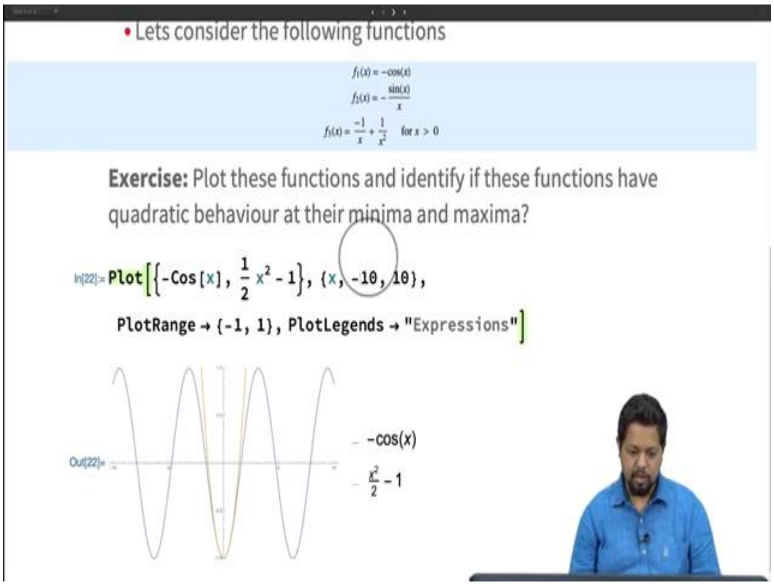

So, you know that for small x, Cos (x) goes like $1 - \frac{1}{2} x^2 + a x^4$ where a is some arbitrary number. So, we ignore what is that x to power fourth order because we are only interested in small x behavior. You see Cos(x) is actually quadratic.

Now, we are interested in not Cos (x) but we are interested in -Cos (x), which means I should multiply a minus sign over there. So, then my result is ½ x^2 - 1. This means that, since the coefficient here is ½ this means this is facing up. This is a bowl that is facing up. So, let us go ahead and plot this function. Since, we have approximated -Cos (x) as this function, let us go ahead to Mathematica back and plot it.

So, this time I will plot it along with -Cos (x). So, I put -Cos (x) inside the curly brackets and another function which is ½ x^2, I think I should go back here. ½ x^2 - 1 and let us see what we got. We need to control the scale. So, what we will do is, it looks like the behavior matches but I will Plot Range I will fix to 0 to 1 because Cos (x) is between -1 and 1. So, -1 to 1.

There we go. It is obvious here that we can go ahead and add PlotLegends. So, orange curve is – ½ x^2 + 1 and you see it so nicely agrees with Cos (x) in this region. We can zoom into the x axis, so that we can see that more clearly by reducing the x range, and there we go.

So, when we say that when angles are small, we can approximate Sin (x) with x and Cos (x) with 1 – ½ x^2. This is what we mean that the function behavior is actually very well captured by the approximated function. So, if I ask you how small should x be, so that you can trust the behavior is like ½ x^2 - 1. The answer to that would be that, that depends on how accurately you are doing things. So, visually here, I can make out differences of the order of for 0.1.

Over here, for example, the difference of the order 0.1, I can make out visually. So, that means if my accuracy is only about 0.1, then this works very well, for x equal to pretty much up to half. So, x equal to half is also reasonably small to approximate Cos (x) with 1 – ½ x^2.

Let us go ahead and try to work out what happens over here. How does the function behave in this region? Is it also quadratic? So, in order to work that out, let us go back to the blackboard. And now we are interested in cosine. We are interested near the maxima and the maxima is at x equal to π. I want to find out the behavior of Cos (x) near x equal to π.

So, for that, I am going to do a Taylor expansion, Taylor expansion says that f(x) is given by f(π) + (x - π) $f'(\pi)$ + ½ $(x - \pi)^2$ $f''(\pi)$ + corrections. In this case, my function f(x) = -Cos(x). So, let us find out what is f(π), which is Cos(π) = -1. So, -Cos(π) is +1 which is what we saw in the picture over here, at this point Cos(π) is -1, so -Cos(π) is +1.

Then at that point we have a maxima. So, f' is going to vanish, if you want, you can go ahead and check it out because derivative of cosine is sine, Sin(π) is 0. So, this term is going to vanish, it does not matter. The next correction I get is from here, which is $(x - \pi)^2/2 * f''(\pi)$. Now, derivative of Cos(x) is -Sin(x), we already have -Sin, so this becomes Sin(x) this should have been x and $f''(x)$ is Cos(x).

So, sine, cosine functions, I take the derivative twice, I get rid of the minus sign, that is what happened. Okay so, I want to find $f''(\pi)$, so that is $Cos(\pi)$ and $Cos(\pi)$ is -1. So, I put in -1 over here, plus other terms, now we are not interested in other terms because we are looking for x close to π, $(x - \pi)$ is very small, $(x - \pi)^3$ will be even smaller. So, ignore that. And let us go ahead and simplify this.

This is 1 plus, expand this out, 1 – ½ also outside I have got $x^2 + \pi^2 - 2 x \pi$. Simplify this, we have got $1 - \frac{1}{2} x^2 - \pi^2/2 + x\pi$. Let us go ahead and plot this function.

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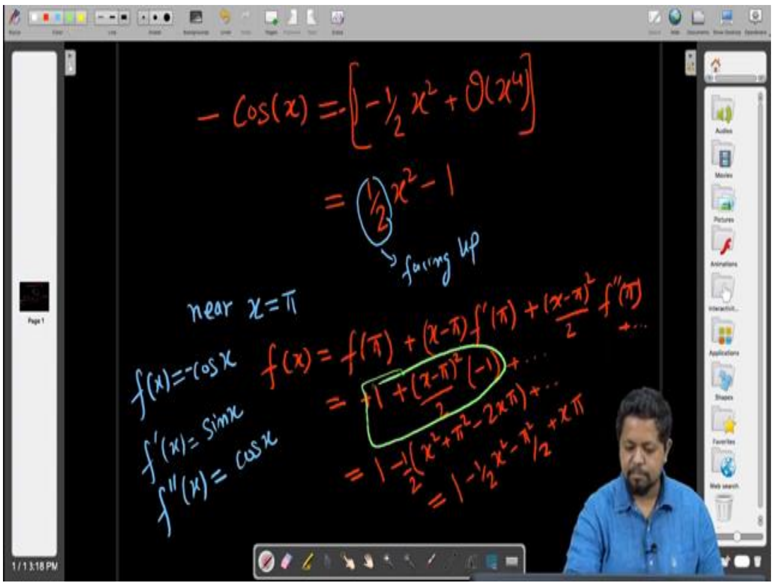

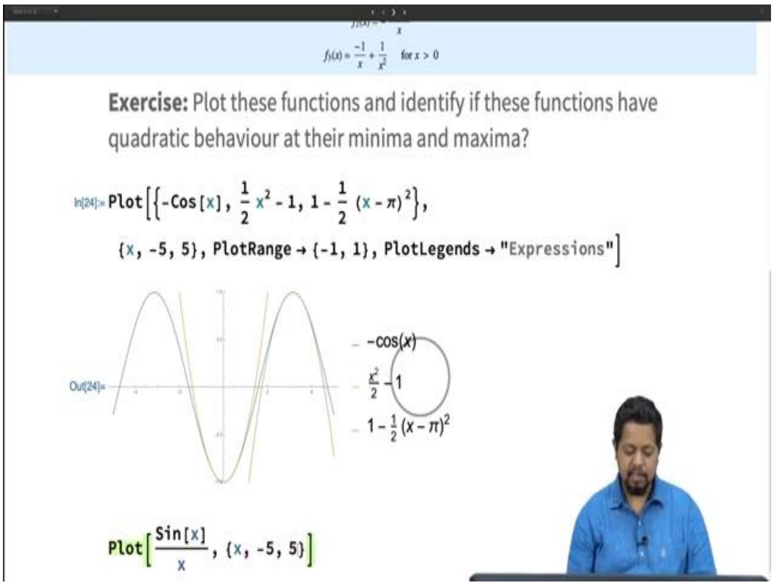

So, alright since I do not need to expand, I can just plot this part to make life easier, what I am going to do is? I am just going to plot this function. So, that is a third function I am going to add to this list, $1 - \frac{1}{2}(x - \pi)^2$ and it will execute that bingo, we get the green curve over here, which is approximation for the function at the maxima and you see that this is also quadratic.

And exactly meets the curvature over here, and that is the reason why we say f1(x) over here -Cos(x) has quadratic behavior at the maxima and the minima because it is a periodic function, the same behavior is going to be repeated at the other maxima and minima. Let us move forward and work on the function f2.

So again, you will start by plotting f2, let us look what f2 looks like, it is -Sin(x)/x. And if you are familiar with this function, Sin(x) vanishes at x = 0, x vanishes at x = 0 but Sin(x)/x does not vanish at x = 0, it becomes 1. So, let me go ahead and put x equal to -5 to 5.

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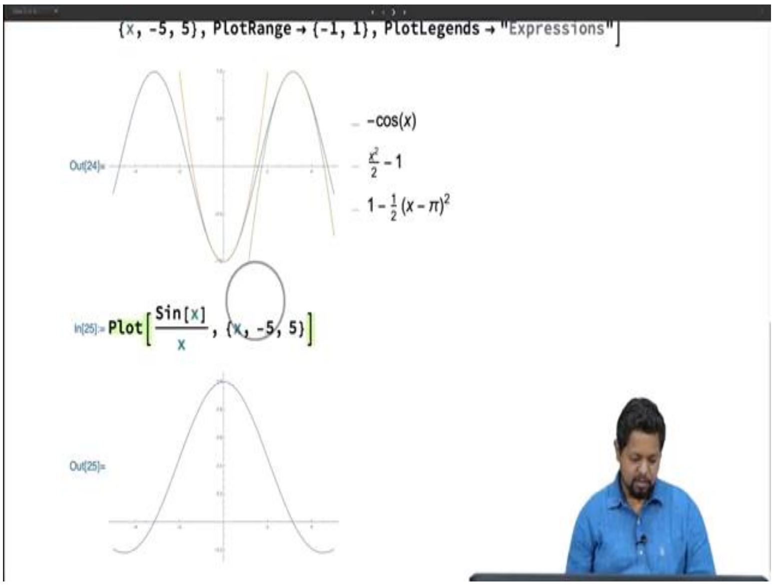
{x, -5, 5}, PlotRange → {-1, 1}, PlotLegends → "Expressions"]
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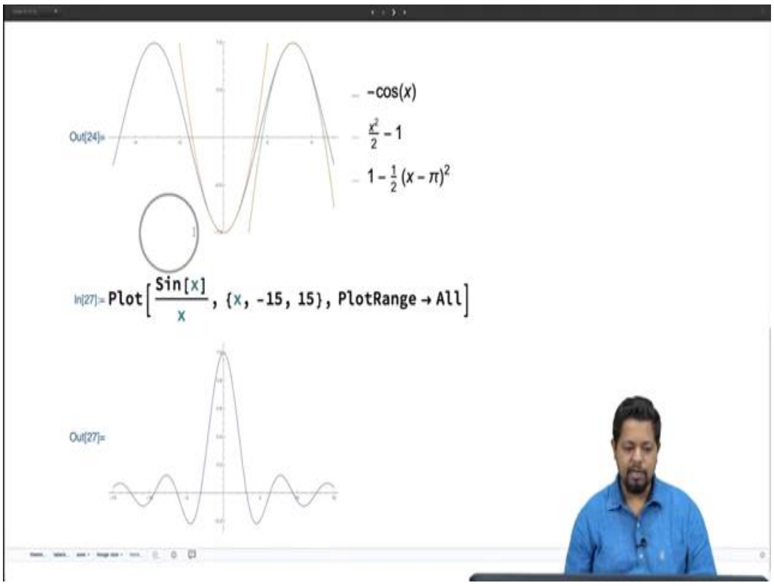
x²/2 - 1
1 - 1/2 (x - π)²
In[25]:= Plot[Sin[x]/x, {x, -5, 5}]
Out[25]=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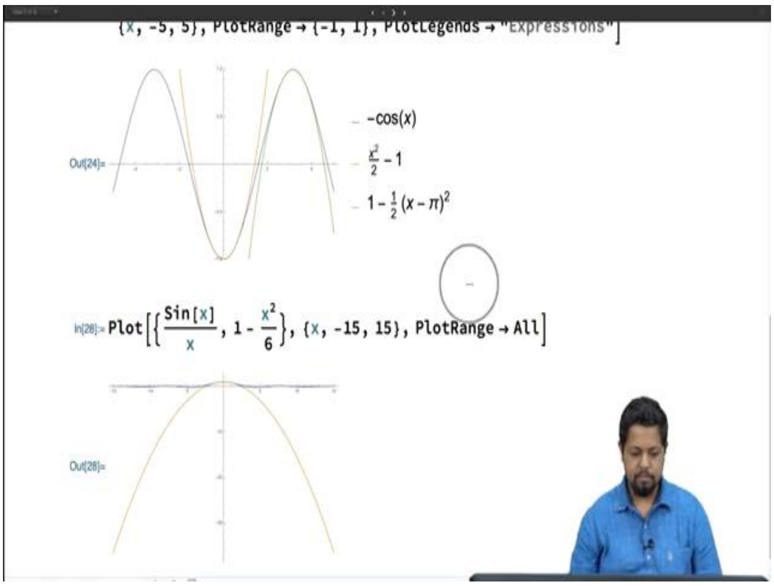
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x²/2 - 1
1 - 1/2 (x - π)²
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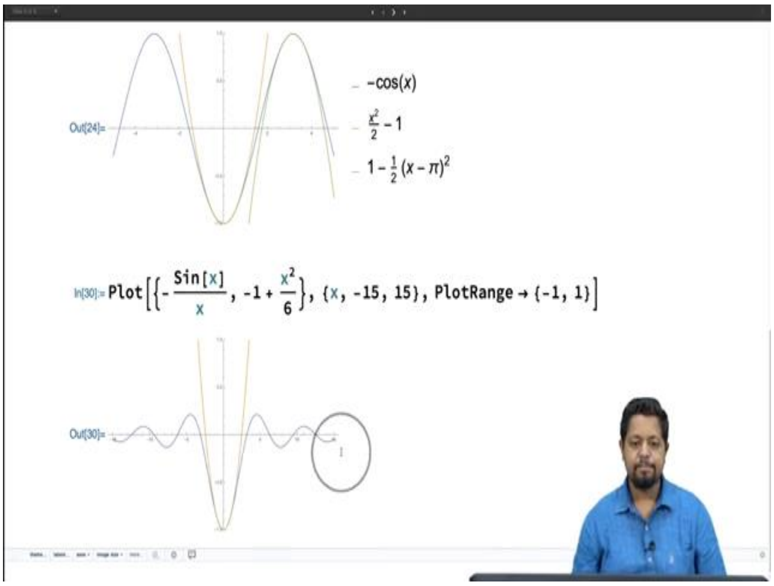

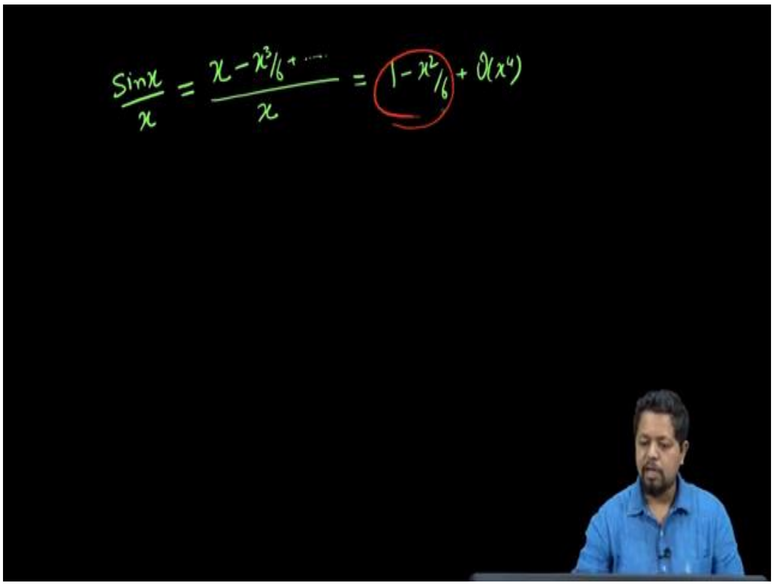

And, again, Sin(x)/x is an even function, we get something like that. Let us go ahead and increase the range. See, how does this function behave? We get that, this is cutting out the peak. So, I will fix that by adding a PlotRange and say All and it will plot everything. Alright, so I have got Sin(x)/x, I see various maxima's and minima's. I want to find out whether the function behaves quadratically over here.

So, how do we do that? We do the same thing as before. We go ahead and approximate the function near x = 0. This time the function is Sin(x)/x, let us first approximate Sin(x). It is 1, sorry this is Sin(x) is x. So, this is $(x – x^3/6 + a\ x^5/5!)/x$, a is some number, I simplify that I get $1-x^2/6$ plus a correction at the order x to power 4.

Let us go ahead and plot this guy and compare with Sin(x) and I get $1 - x^2/6$, again I need to fix the range. So, let me, rather than saying All, I should say, -1 to 1. There we go, we get the agreement over here to -Sin(x)/x, so that is a minimum. So, let us go ahead and do that. I want to do -Sin(x)/x. And I also want to flip the sign of this argument, so $-1 + x^2/6$. There we go, so -Sin(x)/x also has a minima at x = 0, which has a quadratic behavior

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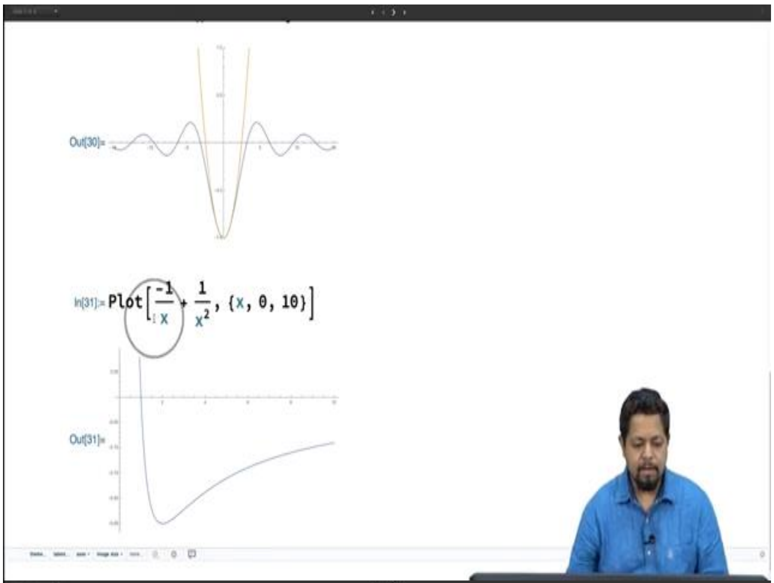

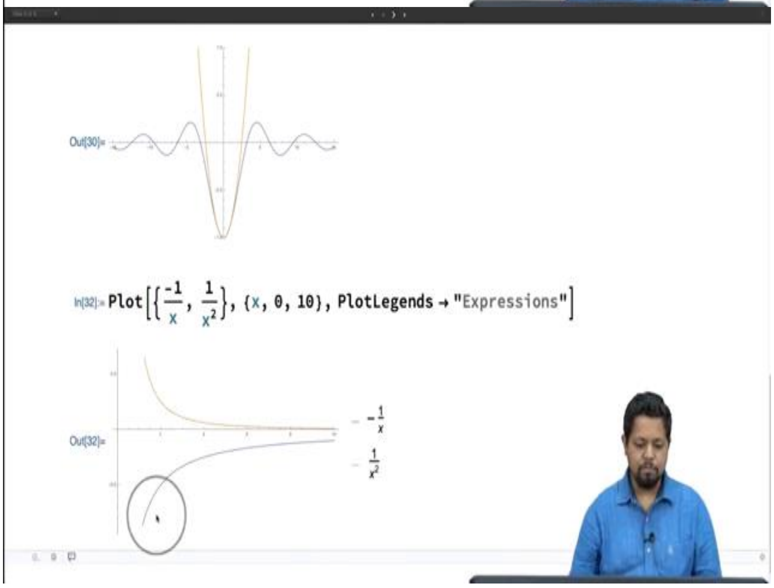

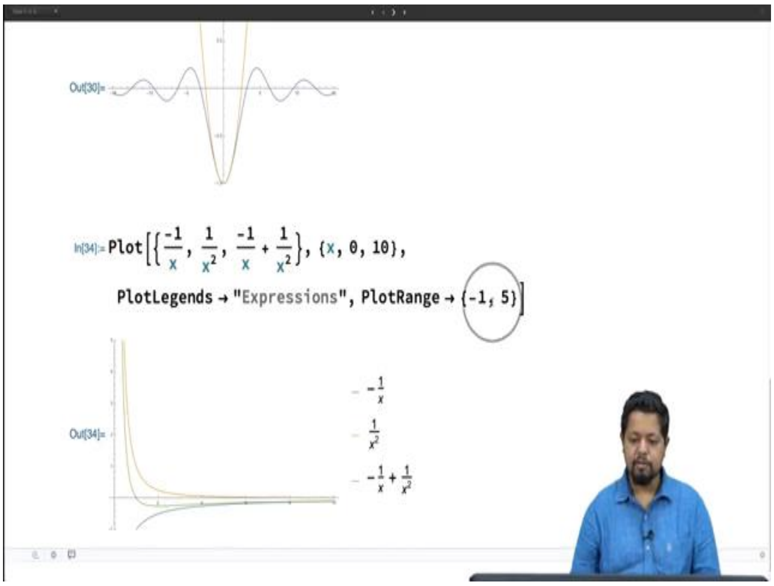

Let us go to the third function, f3(x) which is $-1/x + 1/x^2$. We want to plot this for $x > 0$, so 0 to let us say 10. That is what the function looks like. In fact, it is a good idea to actually understand plotting this function by plotting $-1/x$ separately and $1/x^2$ separately.

So, let us go ahead and do that, we will plot these two functions first. And understand why we anticipate a result like this. So, rather than adding them right away, I will first plot both of them separately, so $-1/x$ and $1/x^2$ and then I will add a PlotLegend. There we go. So, blue curve is $-1/x$ over here and orange curve is $+1/x^2$.

As x is small, you see that $1/x^2$ will dominate, it is going to go like this, this is the dominating curve. So, when I add the orange and the blue curve together. For small x, the behavior will be dominant for the orange curve. So, the addition curve, obtained by adding these two functions is going to go like that curve.

And as you see, as x becomes large, we have more of negative and less of positive, so when we add these two, my function will end up over here. So, it is going to behave more like $-1/x$. So, the asymptotics of the summation function $-1/x + 1/x^2$ is the orange curve for small x and blue curve for large x. So, therefore, and you see that over here it is positive, over here it is negative.

That means somewhere in between it is going to cross 0. So, it is going to cross 0, come like that. And it will end up also having a minima. So, Let us go ahead and on the same plot add the summation function, which is $-1/x$ the sum of the 2 things. There we go, the green curve is the sum of the 2. And as I said, it is going to behave more and more like orange in this region.

And if we say that, you want to plot let us say you want minimum value at -1 and maximum at 5. You see, this is going to go more like orange curve, it converges as x becomes smaller and smaller. That is what we expected.

Let us go back to smaller scales so that we can compare these curves more properly. All right, now we want to understand if there is a minima over here. Does that minima behave like a quadratic function? So, first of all, let us see if there is a minima, yeah it looks like there is a minima somewhere around here. Let us find out and calculate that minima.

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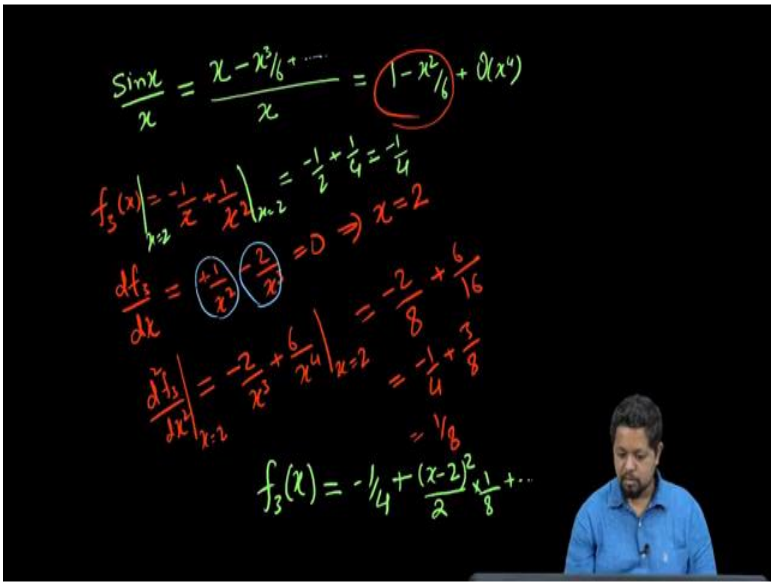

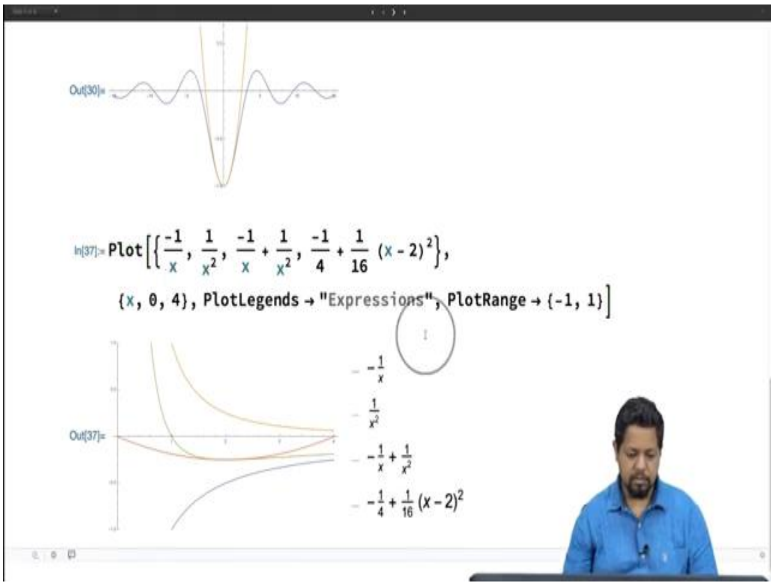

So in order to calculate that minima, we go back to the blackboard. The function that we are talking about here is f3(x), which is -1/ x + $1/x^2$, I want to find what is df3/dx, I get + $1/x^2$ - $2/x^3$, this must be 0 for the minima and you solve this, you get x = 2 as a solution.

So, therefore, I expect the minima to be x = 2 which is what you see in the picture over here. So, minima is at x = 2 let us go ahead and find out what the function f3 behaves like near x = 2. So, let us calculate what is d^2f3/dx^2. So, I have got 2 terms here, I will take another derivative of both these terms and I get minus $-2/x^3$ for this term and then $+6/x^4$ for the next term. And I want to evaluate this whole thing at x = 2.

So, evaluate this as x = 2, which is where my minima is, so on doing that, I get -2/8 over here +6/16. Okay we can go ahead and simplify that and this is going to be -¼ + 3/8, which is -¼ is -2/8 this is 3/8 – 2/8 is plus 1/8. So, therefore, my f3 for f3(x) can be approximated near x = 2 as value of f3(x) = 2 which I can calculate from here at x = 2 is –½ + ¼ which is -¼.

So, it is -¼, the first derivative is 0, so that is gone then I have got +1/8 $(x – 2)^2/2$. So, that is the function I want to plot. There are high order corrections which I will ignore. So, this is the function I want to plot. So, let me go back and plot this, add that over here, this is – ¼ and + 1/16 $(x - 2)^2$ there we go, a new curve has been added to my set of plots, a red curve over here. And you see this matches very well with the behavior of the function near x = 2.

We can zoom into the x axis and analyze it more carefully. So, let me cut it down to four. There we go. So, the red curve is the parabolic curve, a quadratic curve and is approximating very well the green curve near x equal to 2.

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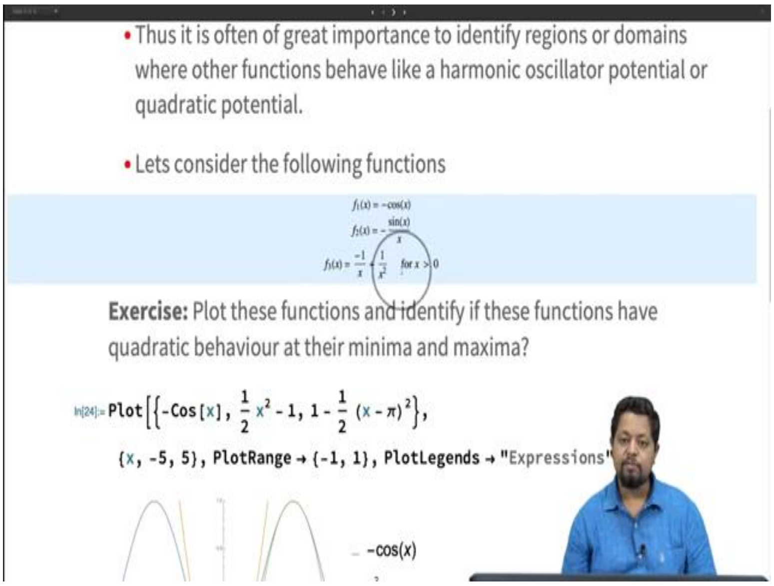

So, therefore, this third example, $-1/x + 1/x^2$ also has a minima. This is the potential that you see in central force potential.

So, this one is an example that comes from central force potentials where you have a repulsive $1/x^2$ behavior from the centrifugal force and -1/x attractive, either a coulomb potential or gravitational potential which goes like -1/r. Ok. So, this was some examples of commonly occurring functions that we see in physics and mathematics. And all of these have at their minimas and maximas have a behavior like a quadratic function.

That is what we mean when we say near a minima we have got a simple harmonic potential. That means near the minima, my function behaves like a quadratic function.