Physics through Computational Thinking
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Module 07 Lecture 38
Linearization 2

Okay. Hello everybody. So, we have been discussing linear systems. And we went into the general theory of linear systems, then looked at a few very interesting examples and looked at familiar examples, you know that we have already seen, but from an alternate perspective. And then we argued for why linear systems, a close understanding of linear systems is good, because of this method of linearization, which can be done even to nonlinear systems. And so, we saw how you know, with the help of a Jacobian you can basically, you know use a linearization scheme and with the help of a Jacobian, say whether the fixed point in question is of a certain kind or not, just from linearization.

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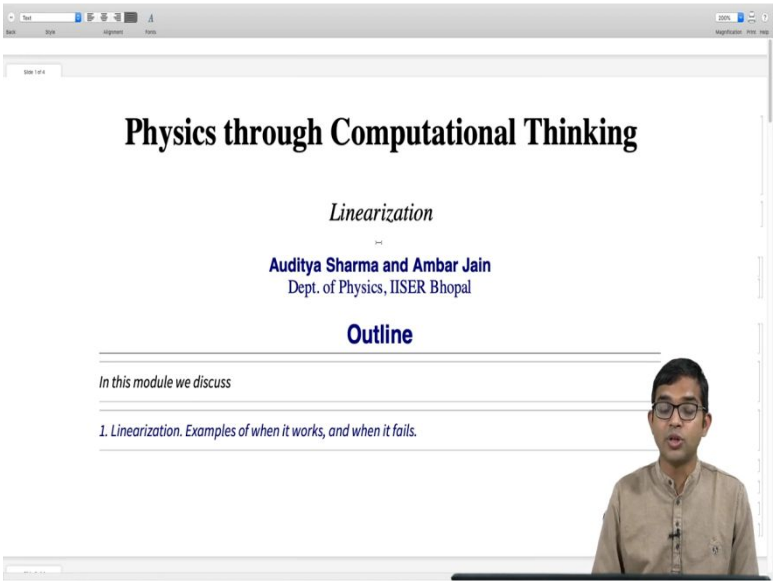

Right. So now what I want to discuss here is, how one should be careful with this, you know. Can you basically treat any nonlinear system as locally linear around a fixed point or is it necessary to exercise some care? So, today I want to show you a few examples, where linearization will work and then I also want to show you how linearization can fail and what one must do to be safe.

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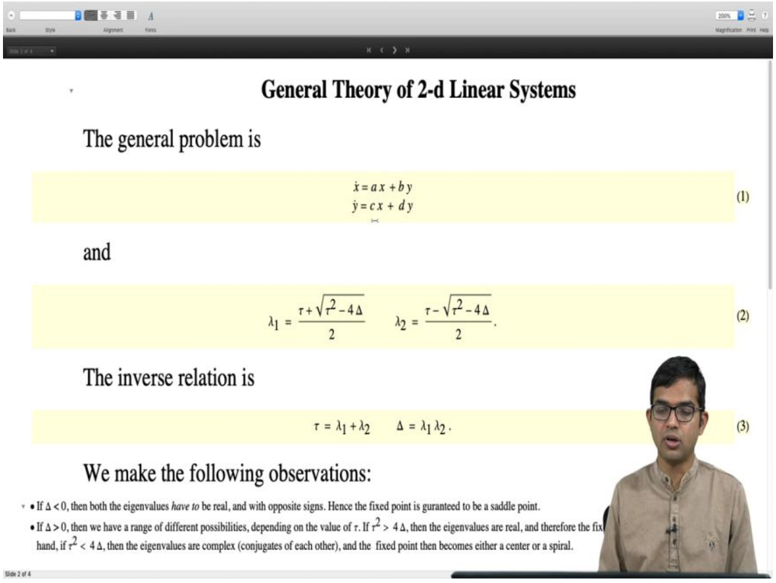

Okay. So, this slide quickly flashes things that we have already seen. So, we have this general theory of linear systems, 2D linear systems, $\dot{x} = ax + by$, $\dot{y} = cx + dy$. And you just simply extract τ and Δ, from a, b, c, d.

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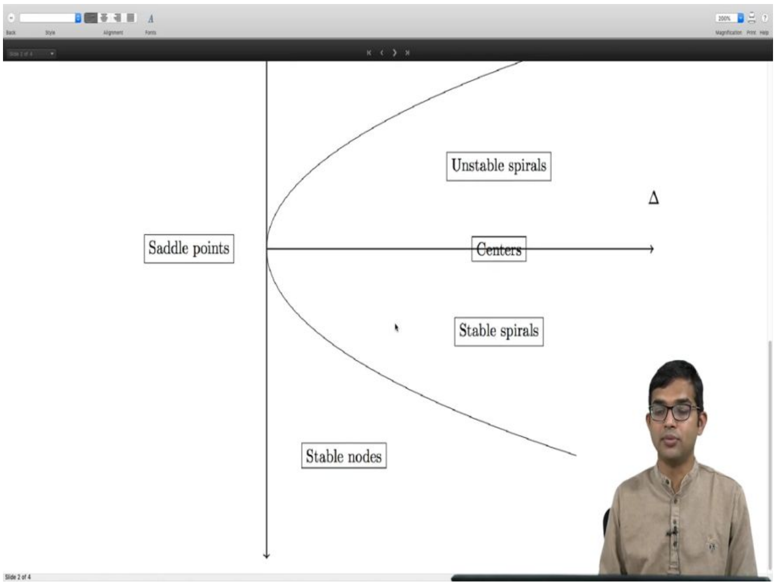

And then depending upon the value of τ and Δ and looking at this picture, you can say what is the nature of the fixed point at the origin. Right?

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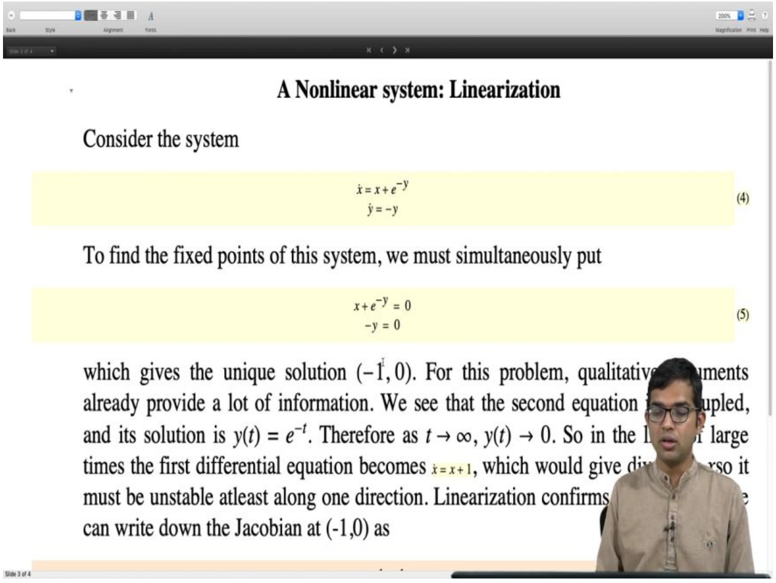

So, then the 2nd step we discussed is that of a nonlinear systems. Right. I mean if you have a general nonlinear system, you find the fixed points in the system and then, and then basically shift your origin to that particular fixed point and linearize around it and solve the linear problem.

Let us look at one concrete example. You have $\dot{x} = x + e^{-y}$ and $\dot{y} = -y$. To find the fixed points of the system, you must simultaneously put, you know both, both the velocities of x and y, or to 0. So, $x + e^{-y}$ must be equal to 0 and $-y = 0$. So, the fact that $-y = 0$ implies that $y = 0$ and you can then directly plug this in into the 1st equation. And then you get $x = -1$. So, the fixed point is $(-1, 0)$. Right. So, we should be able to use the qualitative arguments from before.

Now you see that the 2nd solution is decoupled and its solution is just $y = e^{-t}$. And therefore, as t tends to infinity, $y(t)$ will go to 0. Right. So, in the limit of large times, the 1st differential equation… So, in the limit of large times, you can solve in the 1st differential equation. It just becomes $\dot{x} = x + 1$, because y is going to go to 0. And so, e^{-y} will just go to 1. So, which would give you diverging x. So, it must be unstable, at least along one direction. Right.

So, this is something that we get from intuition. You cannot solve for x as a function of time analytically. But you can basically, already get all the qualitative features, just from this kind of analysis.

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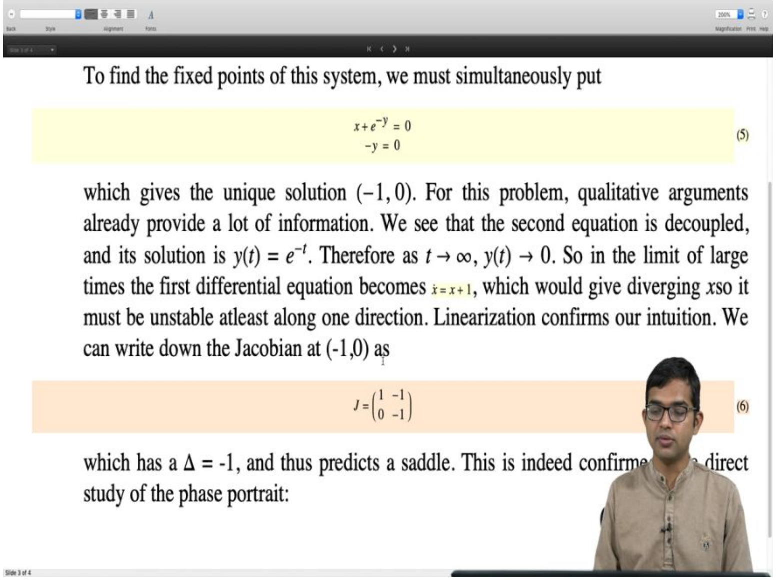

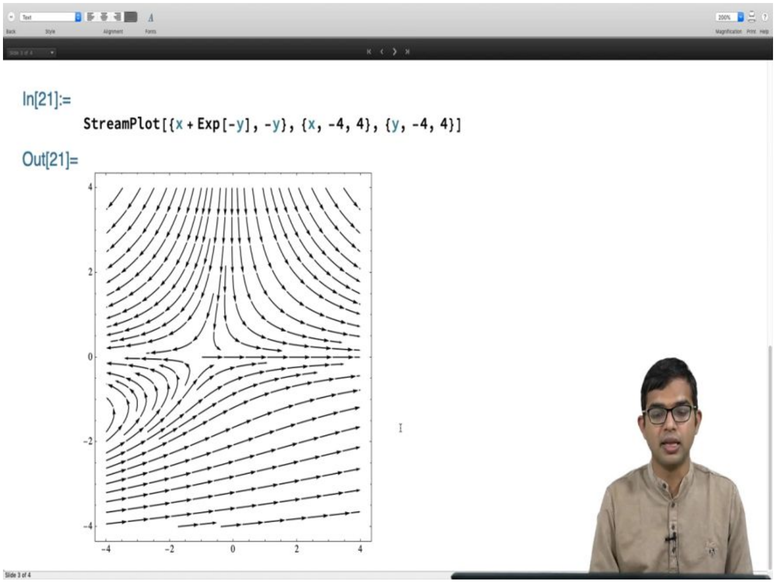

But let us look at what the Jacobian tells us. So, we can write down the Jacobian. And then we see that it has $\Delta = -1$ and thus it predicts a saddle point. And this is exactly, you know matches with our guess. And we can always check out the Stream Plot of this Jacobian. Right. So, this Stream Plot analysis is only for the linearized system. And so, you see indeed that you get a saddle point. Along one direction, it is going to take the system away from the, from the fixed point. And along the other direction, it is going to take your system towards the fixed point. Right. So, the fixed point here is of course at not at the origin, but it is at (-1,0).

Okay. So, so, this is the example, which, where linearization does work. Right; we first analyzed the system, then we linearized it, used the Jacobian, looked at the tau delta of the Jacobian. And we are able to make contact between this Jacobian approach and the full direct analysis approach.

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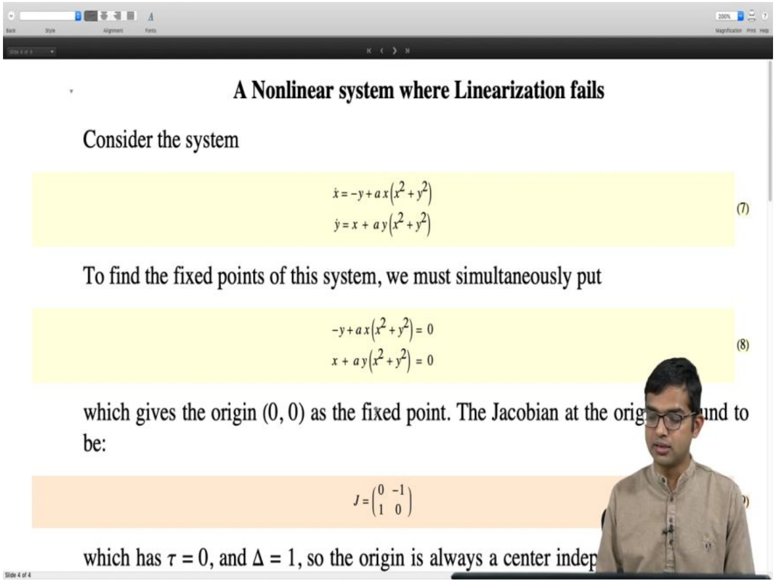

Now let us look at the case, where linearization fails. So, this is just to show you that you should not blindly apply linearization. So, consider this system, $\dot{x} = -y + ax(x^2 + y^2)$, and $\dot{y} = x + ay(x^2 + y^2)$. So, if you put both, both the stuff on the right-hand sides of equation 7 to 0, so then it will give you the origin as a fixed point.

So, the Jacobian at the origin is easy to find out. And what you will find out is, that the Jacobian is basically independent of this parameter a. Right. So, which has just $\tau = 0$ and $\Delta = 1$. So, the origin is always predicted to be a center, regardless of the value of a. That is what our linearization is telling us.

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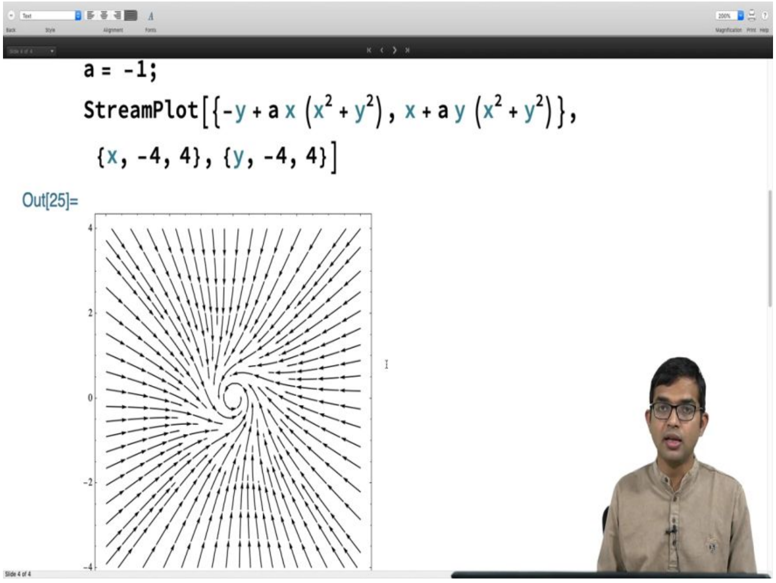

Now let us check whether this is a reasonable finding, by considering a range of values of *a*, and using Stream Plot. Right. So, so yeah, so, Stream Plot you can of course directly use for the full nonlinear problem. Right, I mean you can either use it for the full nonlinear problem or you can use it for the linearized version. But since it is pretty general, so, in this case we want to use it for the full problem itself.

Right. So, a moment ago, in the previous example, I think I suggested that Stream Plot will correspond to the linearized version. But it in fact would work for the full problem itself. And of course, you can also consider Stream Plots of the linearized version. But it actually does not make so much sense. Well, you should use the facility or the function in its full power.

So, let us look at what happens as a function of a. So, suppose I take a to be minus 1 and then, and then I use Stream Plot for this. So, there you go. You see it is, it is an inward spiral. It is going to keep on coming in and it is going to go, it is a stable spiral. It seems like it is going to, all the solutions are attracted towards the origin.

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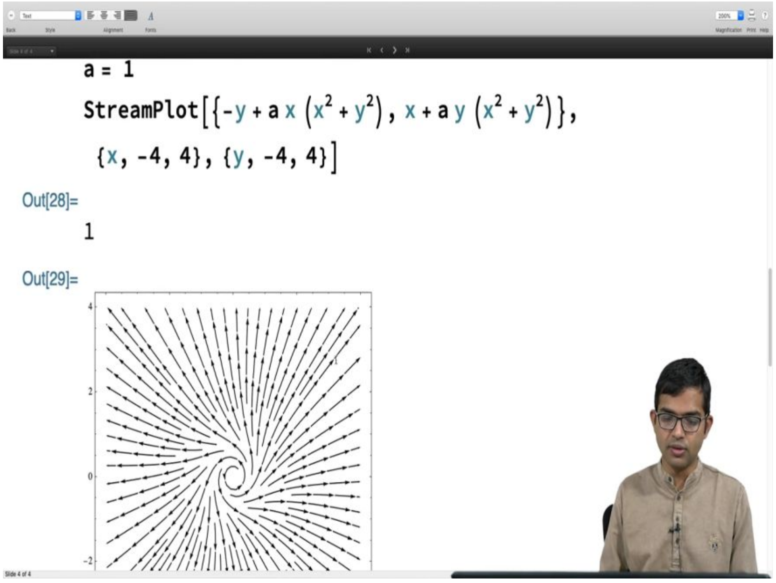

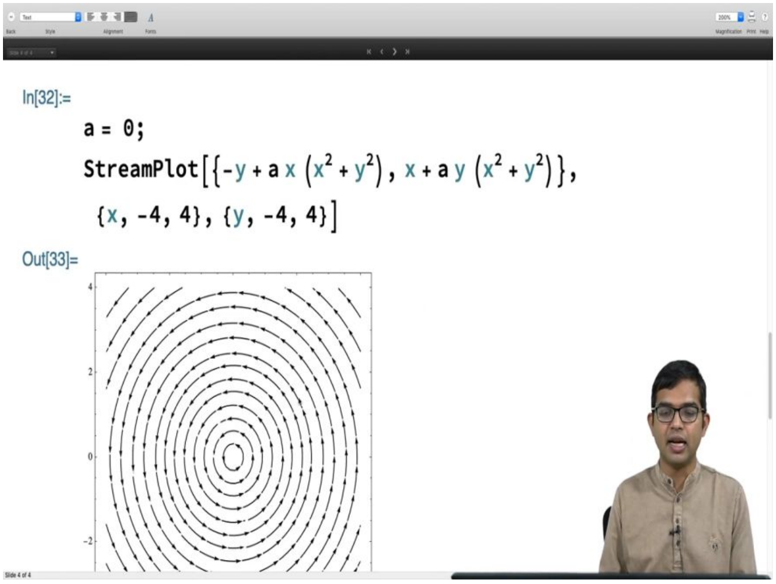

Now but, if on the other hand, if I choose $a = 1$ and do a Stream Plot. I have to do Stream Plot. There you see, it is just the opposite. In fact, it is an outward spiral. It seems like the solutions are all running away to infinity. What has happened? It is like, neither of these is, you know what our prediction was. Our prediction was that it is always a center.

And do we ever get a center at all? So, you should put $a = 0$, you will find that indeed there is a center, which comes about. So, in fact the center comes only for $a = 0$.

And this is not an unfamiliar problem at all, what is $a = 0$. If you put $a = 0$, it is just $\dot{x} = -y$ and $\dot{y} = x$, which is the Humble Harmonic Oscillator problem, the 1D Harmonic Oscillator problem.

And of course, it is going to give you a center. Right. This is like the canonical example of a center is the Harmonic Oscillator problem. It is a fixed point, where there is no loss of energy and then it is neither attracted towards the origin nor is it going to run away. Right? But in fact, it turns out that linearization does fail here.

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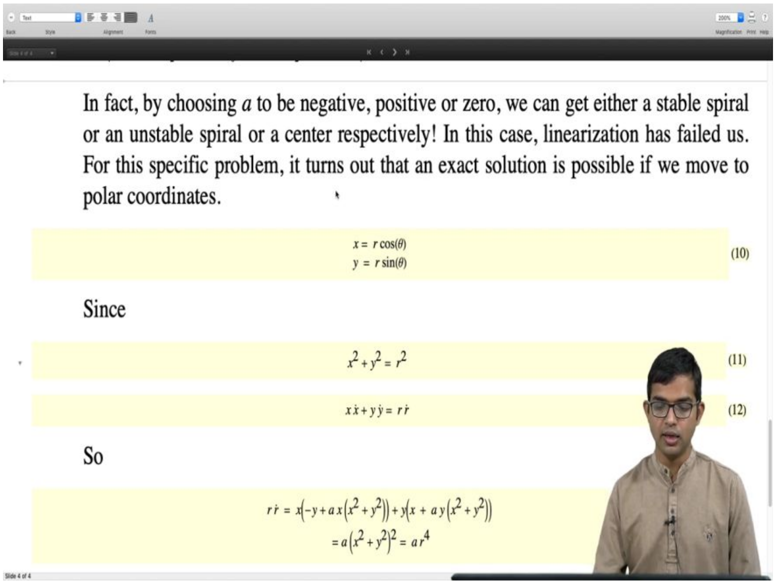

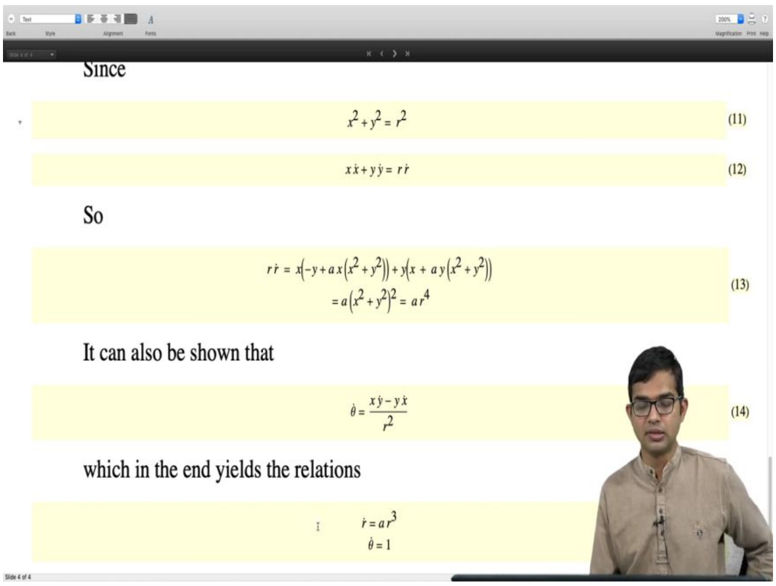

But this problem is one of the rare problems, where the nonlinear system can be solved analytically. All you have to do is, you have to go to polar coordinates. You just shift to polar coordinates, $x = r\,cos(\theta)$, $y = r\,sin(\theta)$. And then you make use of the fact that $x^2+y^2 = r^2$. So, you have $\dot{x}x + \dot{y}y = \dot{r}r$. You should check this very straightforward, but you should check this.

And then, you can look at what happens to $\dot{r}r$. And write it as in terms of, you know x times, in place of $\dot{x}$, you put down the equation for $\dot{x}$. And in place of $\dot{y}$, you put down the equation for $\dot{y}$. And then you do some algebra, you collect terms in a nice way. And then it all comes together in a very nice way. And you have $a\,(x^2+y^2)^2$, which is nothing but $a\,r^4$.

So, you can also show that theta dot is just $(x\dot{y} - y\dot{x})/r^2$. This is something that you should check. Right? And then, which in the end yields $\dot{r} = ar^3$, very simple equation for r, $\dot{r}$. And then $\dot{\theta} = 1$. This is also something you should, you should convince yourself, that this is true.

Right, explicitly put in, you know, in place of $\dot{y}$, you put in stuff from above. In place of $\dot{x}$, you plug in and so on. And do the simplifications, the string of simplifications. And you will get $\dot{\theta} = 1$. So, this is a very very simple set of uncoupled differential equations, if you go to polar coordinates, $\dot{r} = ar^3$.

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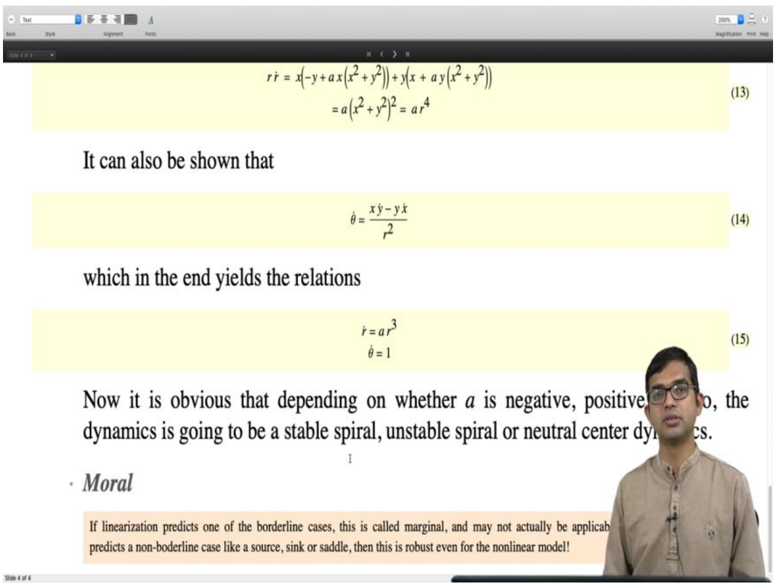

And now it is obvious, that depending upon whether a is negative, positive or 0, the dynamics is going to be a stable spiral or an unstable spiral or neutral central dynamics.

So, the moral of this example, you know doing this example is that linearization is something that you must use with care. And in particular, when you have, when you are sitting on a borderline case, like center is a borderline case. Right. So, is a degenerate node. Right? If you are lying on any borderline of this tau delta diagram, then this is called marginal. Right. So, this is called a marginal case.

And when you are at a marginal case, linearization may or may not work. Right. So, the predictions of linearization are not trustworthy, if you are at a borderline case. But if you are you know, deep inside any of these phases, then linearization will definitely give you the correct qualitative description. Right, it is robust even for a nonlinear model. Right.

So, this is a take-home message from this discussion is, use linearization with care and linearization is not trustworthy, if your system is on borderline case. It may work or it may not work. You need some other way of extracting this information. Linearization is not enough. Thank you.