Physics through Computational Thinking
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Module 07 Lecture 36
Linear systems – Insights from the phase space picture 2

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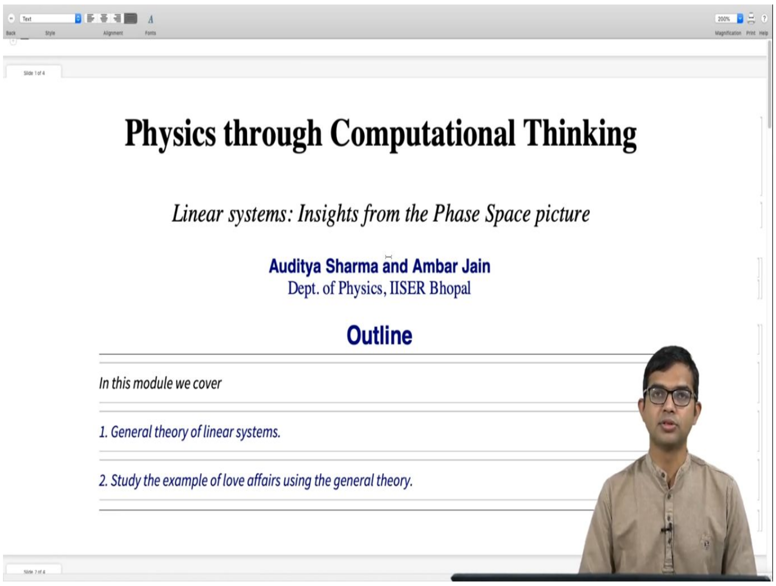

Hi guys. So, in the last video, we looked at linear systems and how you know the phase space picture can give us some very nice qualitative insights. So, then we defined a general linear system and we also looked at how to solve the dynamics, the equations for a linear system in general. But so, let us look at how, you know the general theory of linear systems works and how with the help of the just these two constants; tau and delta, we are looking at two by two systems at this point. So, with the help of just the trace and determinant of the matrix A, which is what we called A last time, you can actually tell, you know the nature of the qualitative dynamics already comes out.

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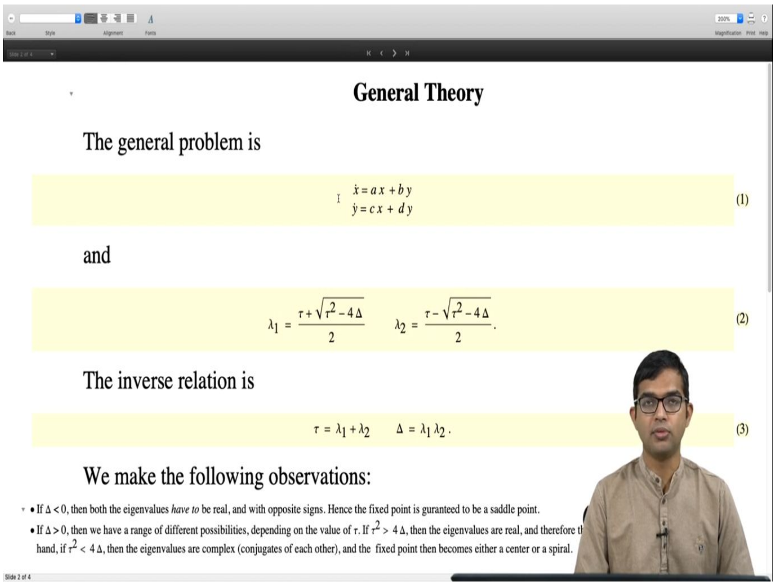

Okay. So, the general problem we have is $\dot{x} = ax + by$ and $\dot{y} = cx + dy$. And we have seen that, the eigenvalues of this matrix *a*, *b*, *c*, *d* are just given by $\lambda 1$ and $\lambda 2$, where $\lambda 1$ can be written in terms of τ and Δ. $\lambda 2$ also can be written in terms of τ and Δ, like here. So, the inverse relation ofcourse is; τ is nothing but $\lambda 1$ + $\lambda 2$ and Δ = $\lambda 1$* $\lambda 2$.

So, the observations we make are the following. Whenever $\Delta < 0$, then both the eigenvalues have to be real and with opposite signs. So, for the moment, let us treat you know, $\lambda 1$ and λ 2 to be real. But I mean in general, ofcourse *a*, *b*, *c*, *d* are completely you know random, real numbers. Even if they are all real, it is not necessary that $\lambda 1$ and $\lambda 2$ have to, have to be real. So, then ofcourse, you will see later on that, it can be extended. These ideas can, will go through even if for arbitrary, you know complex $\lambda 1$ and $\lambda 2$ as in.

For now, let us just think of what happens, if $\lambda 1$ and $\lambda 2$ are real. If $\Delta < 0$, then λ, both of them have to be real and with opposite signs. Hence, the fixed point is guaranteed to be a saddle point. So, you know what happens in a saddle point. So, once again I am using the notion of a fixed point without having defined it. But I think it is intuitively clear that what we are doing is that we are looking at the dynamics around the origin. For a linear system of this kind, the origin is always a fixed point.

A fixed point is one, where the dynamics basically is, there is no dynamics. If you are, if you are stuck at the origin, you will be there forever. But the question is, if you happen to move slightly away from the origin, will you return to the fixed point or will you run away or will you selectively return to it, if you are taking certain special direction. If it is a saddle point, it means that one of these eigenvalues involved; $\lambda 1$ is going to be positive and the other one is negative.

So, if it is the negative eigenvalue, we will tend to bring it towards the origin and the positive one we will tend to take it away from the the origin. And this is what is called a saddle point. If $\Delta > 0$, so, basically if $\Delta < 0$, it does not matter what, where tau is. It is always going to be a saddle point. But on the other hand, if $\Delta > 0$, then we have a range of possibilities. So, now ofcourse, yeah; now it depends on the precise value of τ. You can have real or complex or whatever. If $\Delta < 0$, then for sure, in fact you are guaranteed that λ's are going to be real.

And they are going to have… so, that comes from just looking at $\lambda 1$ and $\lambda 2$. So, that is what, that is the case we already considered. If $\Delta < 0$, then $\tau^2 - 4\Delta$ is also going to be, is necessarily positive. So, basically $\lambda 1$ and $\lambda 2$ will be; one of them is positive and the another is negative. That is for sure. And if $\Delta > 0$, then depending upon τ, you may have $\lambda 1$ and $\lambda 2$ be complex conjugates of each other or they may be real, right. If $\tau^2 > 4\Delta$, then the eigenvalues are real. And therefore, the point, the fixed point is a node.

A node is where you know your, it can be a stable node or an unstable node, we will come to that in a moment. And a node is where you know, in both the directions, the system is basically behaving in the same way. It is both; both the directions are trying to bring it towards the origin or away from the origin. We will discuss in a moment. On the other hand, if $\tau^2 < 4\Delta$, then the eigenvalues are complex conjugates of each other and then the fixed point can become either a center or a spiral. What that means, we will see in a moment. We have already seen a center. Spiral; we will see some examples of that.

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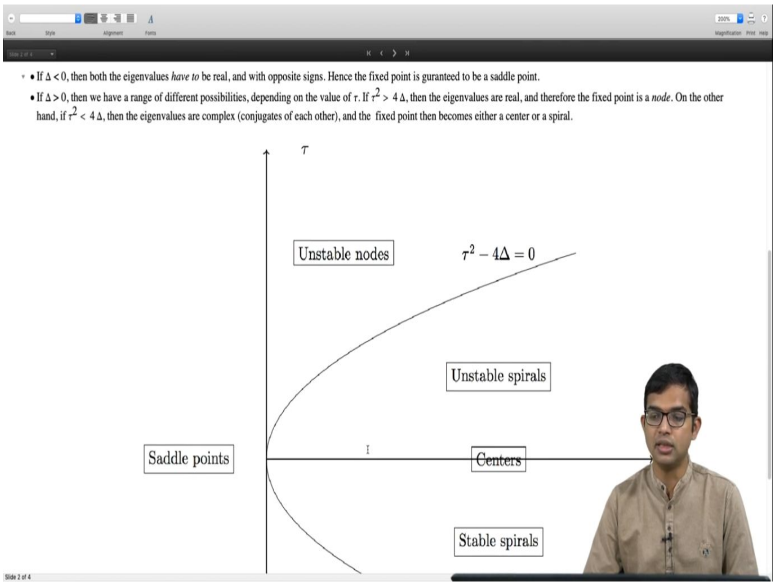

So, this is a compact way of representing this. So, along the Y axis you have τ and along the X axis is Δ. So, if $\tau^2 - 4\Delta = 0$, you are sitting on this parabola. And if you are inside this parabola, then you have, you know depending upon whether your τ is positive or negative, you will have unstable or stable spirals, for sure. And if if you are beyond this parabola and depending upon whether you are above, you know in the… for positive Δ , you will get unstable nodes and for negative Δ, negative τ, for positive τ you get unstable nodes and for negative τ, you get stable nodes, right. So, this is the theory.

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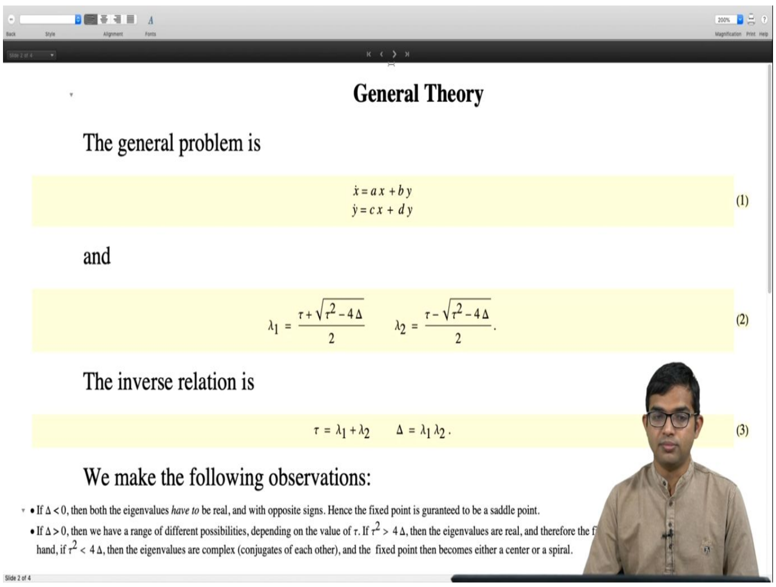

Now we want to see whether we can use this theory, to understand some examples. So, let us look at some very simple examples first and then we will go to Strogatz Romeo Juliet problem. Right; so, let; now let us investigate what happens for $\dot{x} = y$, $\dot{y} = -2x - 3y$. So, τ is just simply -3 in this case. And Δ is just 2. You can quickly calculate it. And so, the expectation is what? So, since Δ is positive and τ is negative, we should look at $\tau^2 - 4\Delta$. τ^2 is 9 - 8 is 1. So, you have $\tau^2 - 4\Delta$ is positive. And so, the expectation is that it is going to be stable node.

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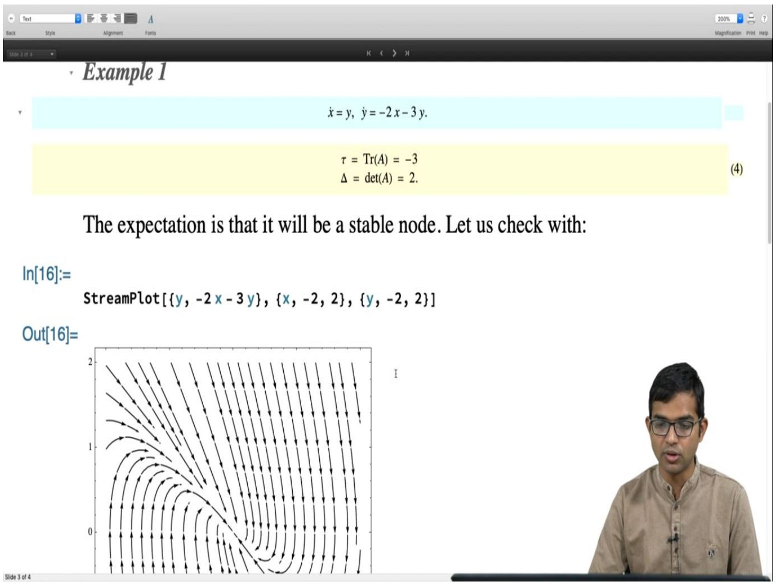

So, let us see what that means, when I use Stream Plot. So, you see that it is stable. So, you see that all curves are basically coming towards the origin. And it is a node. Right; it is along both the directions. So, the directions in this case turn out to be like very sort of close to each other. And that is why it seems a bit messy; this figure. But basically, it is just that both the directions are attractive in nature. And so, all these trajectories will just take you to the origin.

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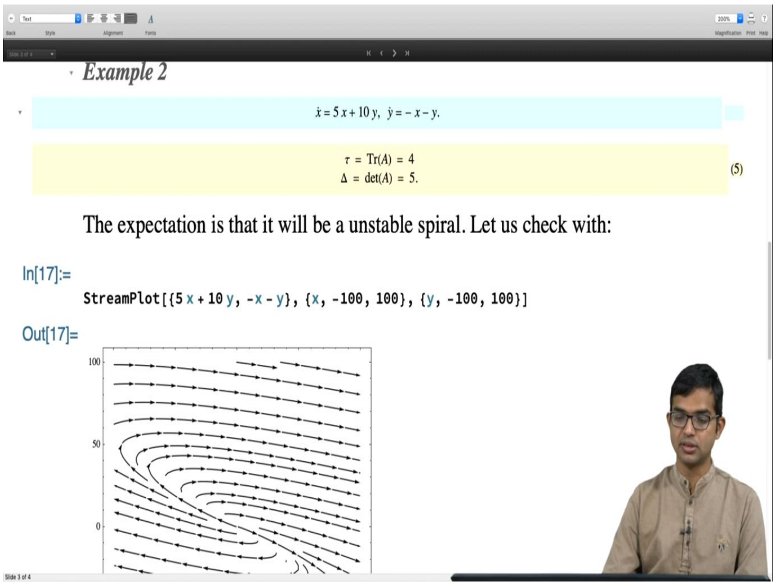

Now let us look at another example. We have $\dot{x} = 5x + 10y$, $\dot{y} = -x - y$. Now your τ is 4 and Δ is 5. So, τ^2 = 16 - 20; -4. So, this is going to be, since it is negative, you have *a*; and your Δ is positive, so, the theory tells us that we should look for a unstable spiral.

Let us see what that means. So, unstable spiral; so, you see that there is; yeah, so, it is an unstable spiral, it is not very clear on this scale. Maybe you should zoom it out a little more and then you will see that, you know there is a continuous loss of energy in an unstable spiral. And then it will keep on going away. So, one thing that is evident here is ofcourse that, solutions tend to run away from the origin. It is unstable in both the directions and it is not going to just go in, you know monotonical, it is going to keep circling around this and then it will run spiral away to infinity. That is what is an unstable spiral.

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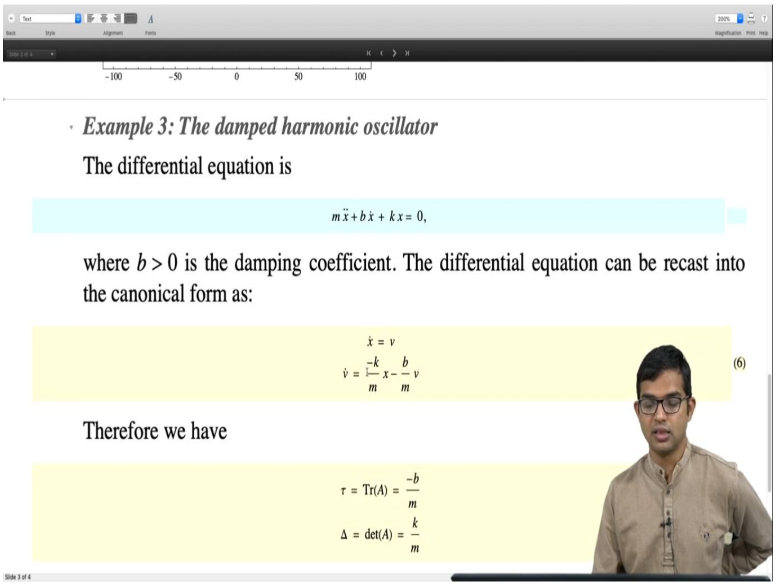

Let us look at the familiar example of the Damped Harmonic Oscillator. Right; so, the differential equation is; and from within this theory, we all know how to solve this. We have looked at this from multiple directions. We have also looked at how to solve this numerically with RK4 and you know even simpler methods like Euler and Improved Euler and all this. Now let us see how to analyze this from this general linear theory point-of-view.

So, $m\ddot{x} + b\dot{x} + kx = 0$. Now ofcourse, we want to bring it into this vector form, canonical form $\dot{x} = v$ and $\dot{v}$ is equal to, you know this is linear combination of x and v in this manner. So, we can rewrite this as, you know exactly in the form we had earlier and from where we can extract the τ and Δ. τ of this trace is just nothing but the trace of A, which is $-b/m$. And Δ is equal to determinant of A, which is k/m.

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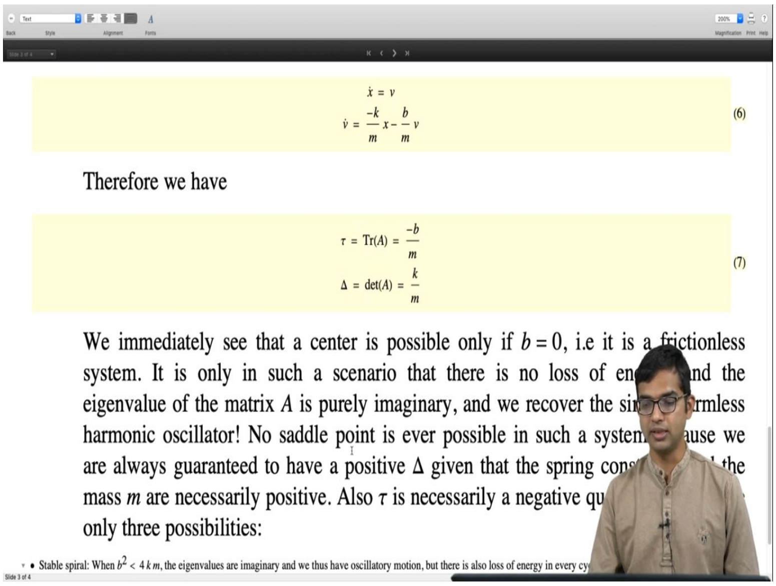

So, we immediately see that a center is possible, only if $b = 0$. Why is that? A center; for a center to happen, we have seen that your tau must be zero. And so; and that is the case, where you have a frictional, a frictionless system. And that is the simple harmonic oscillator. But if b is non zero, there is going to be, you know no energy conservation. You can have a fixed point, but it is going to be certainly not a neutral fixed point.

You will see that there are 3 different cases, which arise. You know, because for sure, you have to be, your Δ is positive. So, there is no way for you to tune your Δ to be negative, because you have a mass and this constant k, which is sitting on the right-hand side and both of these are necessarily positive quantities. Unless you have some weird spring, where you have, you know with a negative k, which is very very unphysical. But maybe there is a way to artificially set up such a system, but we are not really going there.

So, k is definitely a positive quantity for all genuinely practical purposes. So, we will take k to be positive and then Δ is guaranteed to be positive. So, there is no question of a saddle node for, for these kinds of systems. So, the only thing to decide is whether you are inside the parabola or outside the parabola, or if you are on the boundary. If you are on the boundary, you get this borderline k, which is called as degenerate node and which is what is called Critically Damped Motion.

So, we see that, you know the physics of damped motion, critically damped, underdamped. So, it is underdamped, if your, if $b^2 < 4\,k\,m$ and the eigenvalues are imaginary, they are conjugate imaginary eigenvalues. And if $b^2 > 4\,k\,m$, the eigenvalues are real. And there is a mixture of growing and decaying motion, but with no oscillatory motion. So, it is Over Damped motion and there is continuous loss of energy.

And just there is not enough time for the system to execute even one oscillation. So, the damping is so heavy that it will keep on falling down. And then it will eventually go to the, it gets absorbed by the, into the origin; which means that it comes to rest. Okay. So, this is; once again it is useful to look at the Stream Plot of this. And so, you see that depending upon; so, what have I got here; $-x-2y$.

Well, I have chosen some particular values of this. So, this is something that you should play. You should play with different possibilities for b/m, k/m, τ and Δ. And see for yourself that, you know all these 3 cases, which are familiar territory, but now from an alternate perspective. That is, that is going to be an exercise for you to carry on.

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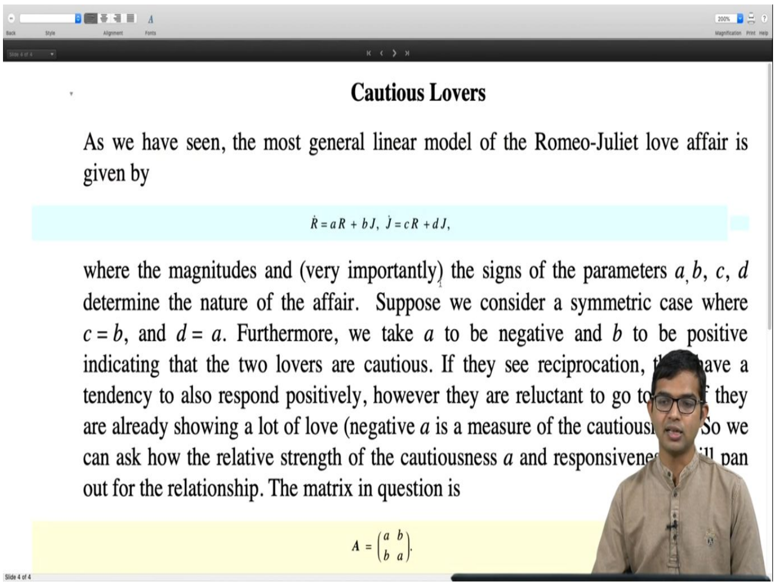

Let us move on and let us look at this one, another variant of this Romeo-Juliet problem of Strogatz. So, $\dot{r} = ar + bj$ and $\dot{j} = cr + dj$. This is the most general linear system of Romeo-Juliet, that you can come up with. But suppose you take a symmetric case, where

$c = b$ and $b = a$. And furthermore, we take a to be negative and b to be positive. So, basically this is what he calls cautious lovers.

So, if the c reciprocation, so then they have a tendency to respond positively. But on the other hand, they are reluctant to go too far, because they are afraid that they may get hurt. If they are already showing a lot of love, then they want to be cautious. So, this is the relationship. And then you can ask; what happens? What is the fate of this relationship? So, you write down the matrix. So, A is negative and B is positive. So, the matrix is just, you know like here.

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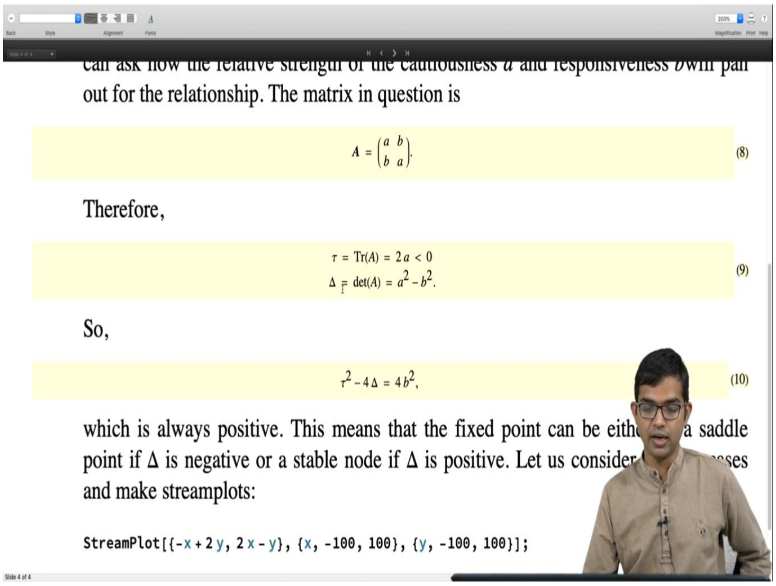

So, all we have to do is find out τ and Δ. Trace is $2a$ and the crucial thing is that $2a < 0$. That you should keep in mind. And Δ is $a^2 - b^2$. So, $\tau^2 - 4\Delta$ is $4b^2$. So, this we know is an important quantity. If $\tau^2 - 4\Delta$ is greater than zero or less than zero or equal to zero, you can have different kinds of physics, which comes above.

So, this means that the fixed point can either be a saddle point. If Δ is negative; if Δ is negative, then you do not even have to look at what τ is. If Δ is negative, what does it mean? It means that $a^2 < b^2$. And then you are not too cautious. If $a^2 < b^2$, then perhaps it is good for the relationship. So, let us see how that plays out in a moment. But on the other hand, if $a^2 > b^2$, if Δ is positive, then you get a stable node.

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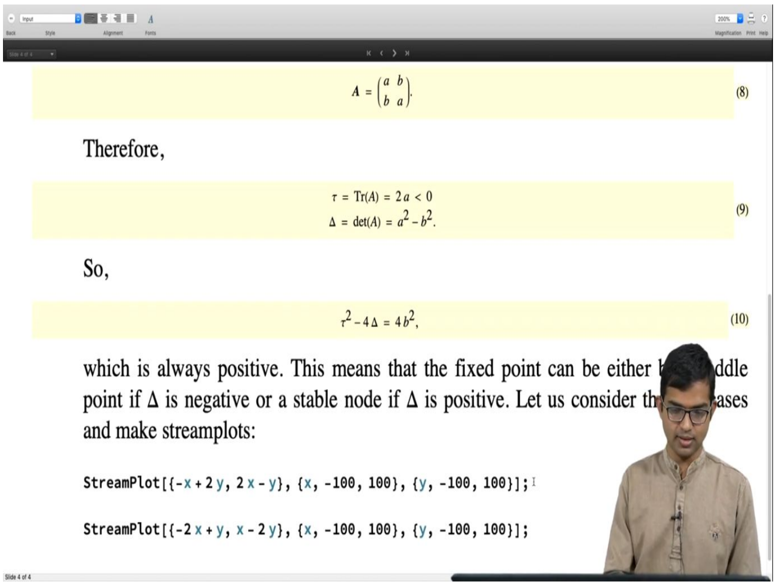

So, let us look at the Stream Plots of both of these cases. So, you can convince yourself of this, by looking at these equations for a moment and imposing the condition that, *a* is negative. So, that is an important condition. So, let us see what happens in the 1st case.

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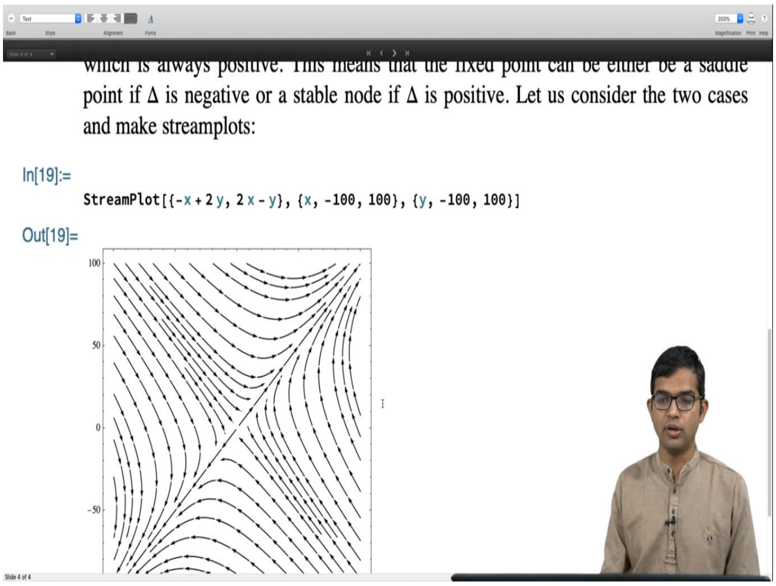

So, you have a saddle node, if delta is negative. I have chosen delta to be negative here. And then what happens is, basically they are not so cautious. So, in this case, you can, depending upon the initial conditions, it can, you know, take you to the, to a very high positive value, both, *x* and *y*. So, in this case *r* and *j*. Right; *r* is along I think the X axis and *j* is along the Y axis. It does not, whatever, you can check this. So, you will find that, in the 1st quadrant, if you are along any of these initial conditions, such that you will be attracted towards the $r = j$ line, that is a happy sign for the relationship.

Basically, what it means is that it is going to keep on increasing in value and they are going to have a love fest of some kind. But on the other hand, if your, if your initial conditions are somewhere in this region, in the 2nd quadrant or in the; sorry, in the 3rd quadrant or in the 4th quadrant, then you can basically get attracted towards r equal to, $r = j$, but in the negative direction. And so, that means that you have very large negative values of, you know both these; feelings of both of them will be very large negative numbers.

Ofcourse, if you happen to be on this very special line; you know y equal to minus x, then it is just going to come down to the origin. And then the relationship will fall apart. But in general, you see that this relationship has an explosive nature to it. It can be; depending upon the initial conditions, it can be really heavily positive, or heavily negative. That is one type of solution, which is possible.

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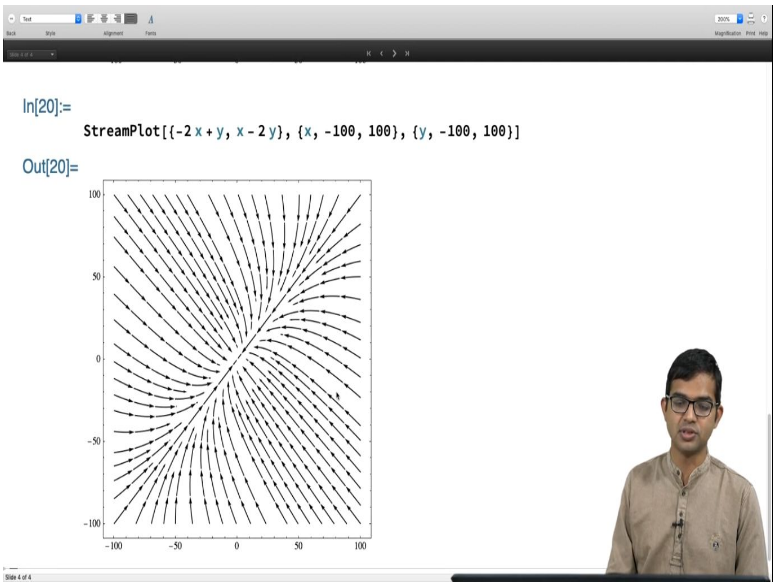

And the other type of solution, which is possible is this one; which is, which is kind of a sad thing; which is like, when they are too cautious. And over-caution is apparently a bad thing. If *a* is large; right, if *a* is large, then your delta is going to be positive. And that is not so good for this relationship. It is going to become a stable node. So, no matter where you start, it is going to fizzle out. And then eventually, both; r and j will just go to zero. So, it is an unhappy end to this relationship.

Okay. So, you should play with these coefficients; a, b, c, d. Right; you know these parameters, rather of this model and see if you can come up with more interesting possibilities. And think about, you know creative ways of building on this model and and play more. Okay, thank you.