Physics through Computational Thinking
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Lecture 34
Driven Oscillation: Variation

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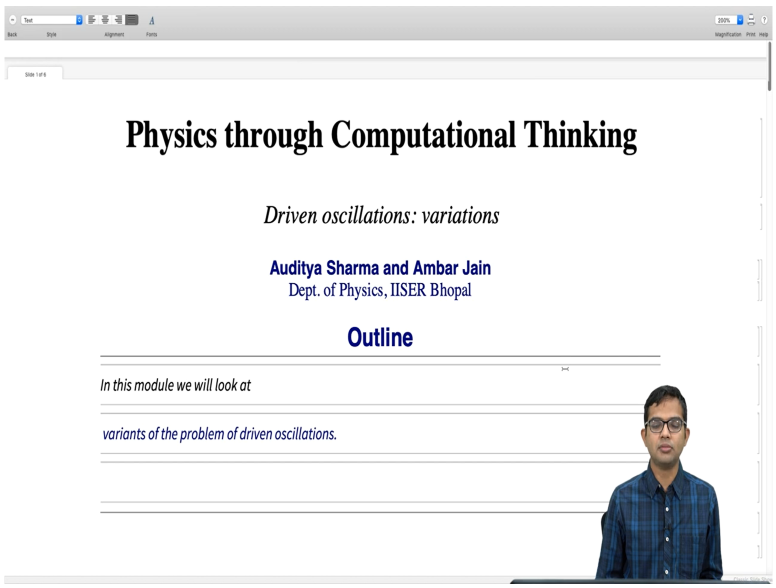

Hi guys, so this is a follow up module to driven oscillations which we have already looked at and so what I want to do is show you that with the help of some small tweaks around the problem we worked out. There are many flavours which come out so this is within the philosophy of this course which is to explore you know to build a model or a method which is solid.

Take very simple decisive solid steps forward. And then use this opportunity created by you know something that you have lead the foundation for and do a small exploration around it. And then you see that so many colours you know come out you know this kind of a, an exercise.

So you can think of this as a you know tutorial and I would challenge you to pause at various point and actually execute everything on your own and not just blindly or passively watch me go over my just only do not allow me to passively talk through. So I would expect that you pause at various points and then workout your own method and come up with the full analysis and then watch my video. And hopefully you will have more perspectives and it can all be complimentary in nature.

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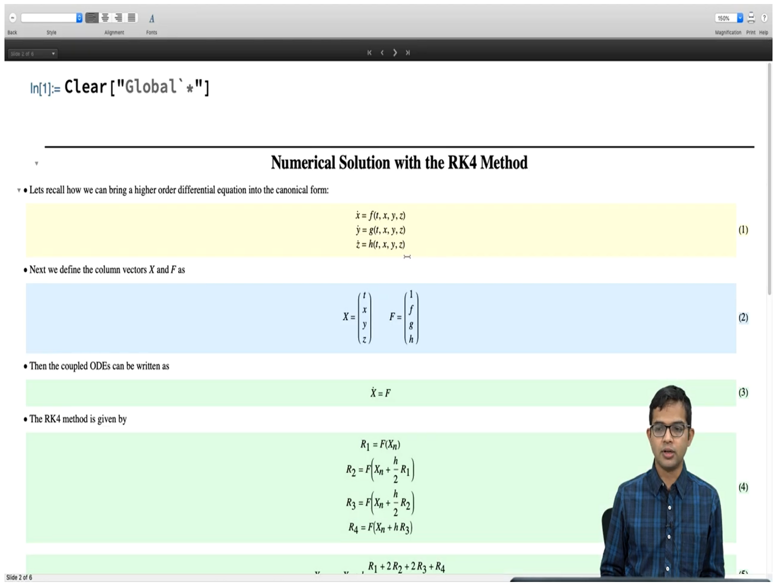

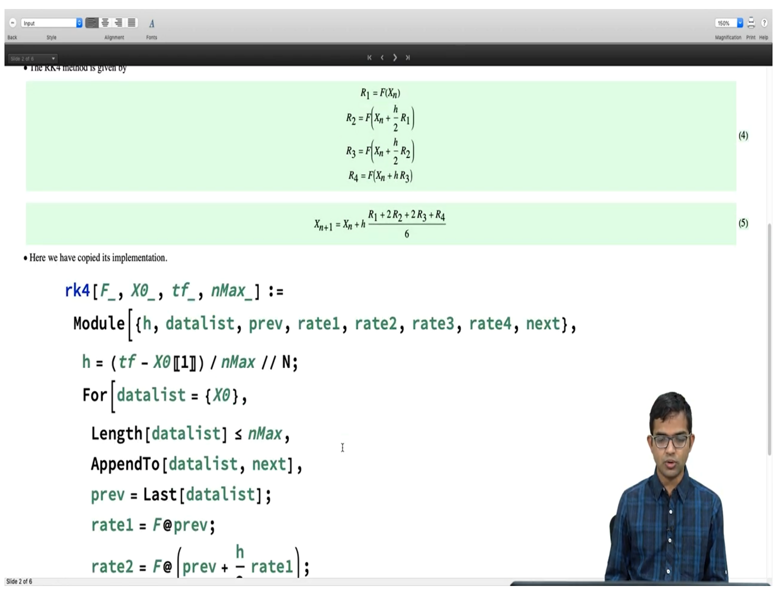

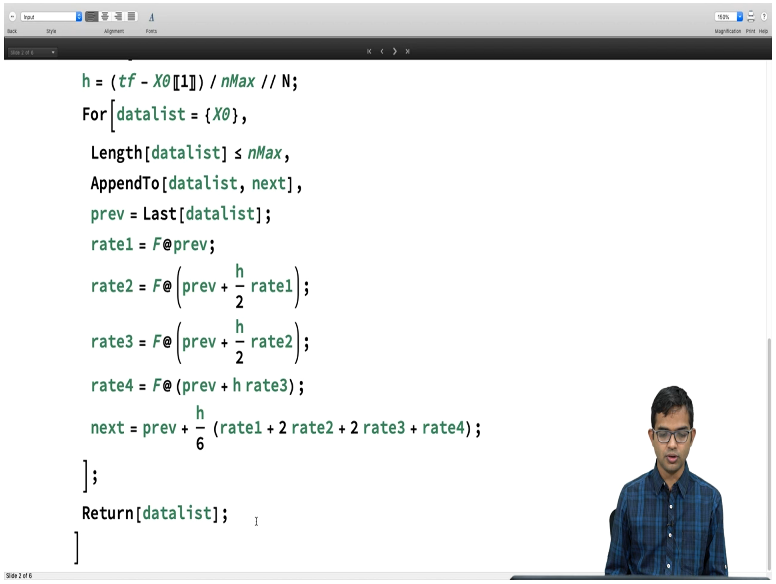

Alright, so as always it is useful to start Mathematica session using this clear which I have just done already. And I am going to quickly flash this RK4 method slide you must have seen it before. So basically the idea is that you first rewrite your equations in this canonical form. And so there is very nice generic formulation which we have in terms of vectors where you basically include time also in to your x vector.

And then you rewrite your differential equation involving as many variables as you need and of whatever order you have to first recast an arbitrary order equation into a first order equation involving more variables. So that is a first step and time itself is treated as one variable so when we do this we are able to rewrite it as $\dot{x} = F$.

So we saw that the Euler method is the simplest you know where x dot is approximately taken to be the rate at that particular instant of time alone and then the Improved Euler took an average of the velocity at that point. And the following point and took an average that itself gave a substantial improvement and this is equivalent to the RK2 method.

And then RK4 method is a, you know, more sophisticated method and turns out to be an excellent approach all together because it nicely balances the twin aspects. You know there is a competition between the number of operations you do not want too many operations on the one hand but also you want accuracy high accuracy and minimal operations.

So these are contradictory forces somehow experience told us that RK4 seems to be the most acceptable accepted compromise between these two opposing forces. Okay so I am going to just load this code so the details of this code you can find in an earlier video you can go back and check it out. If you are interested for our purposes we just simply load this code. So what a physic problem I want to look at.

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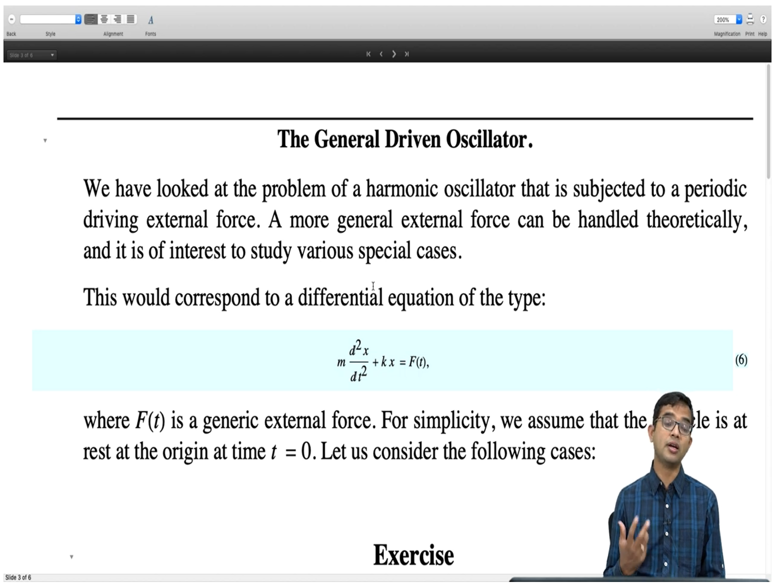

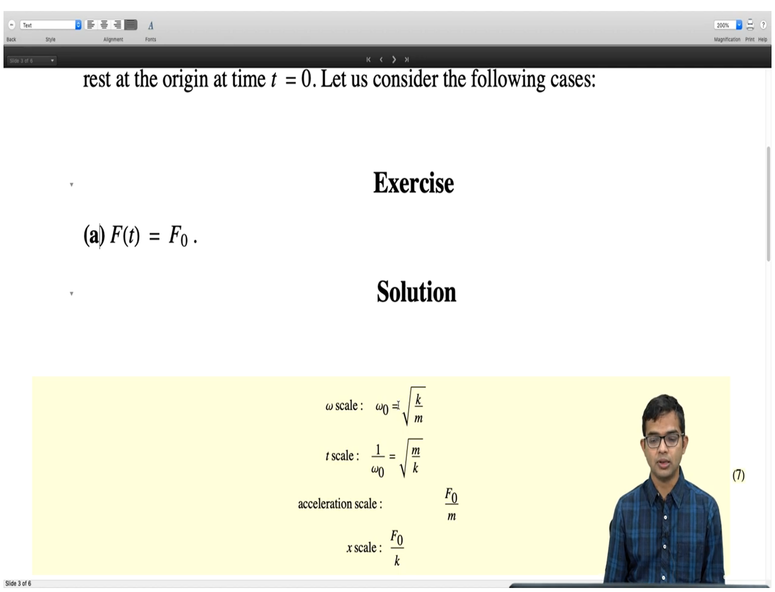

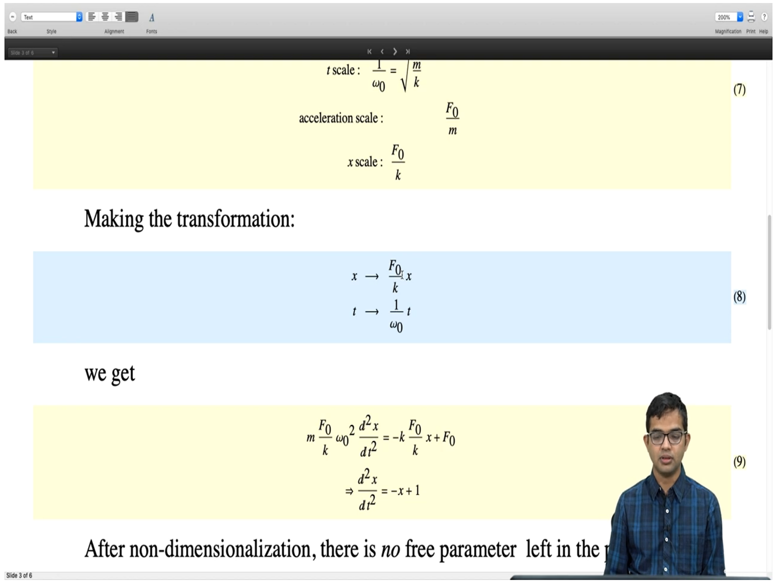

So physics problem is a variant of the driven oscillator problem that we have looked at earlier. But I want to consider a bunch of new cases here, well so we have looked at that is subjected to a periodic driving external force. It is not just a driven external force but we have looked at the one particular case which had which was periodic in nature.

And then, we looked at the special case at what happens at resonance when the frequency of the external driving force was equal to the natural frequency of problem in question. But actually it turns out that it is of interest to look at more general types of external forces. So you could consider a much more complicated external function and it does not have to be periodic in nature or it may have a portion which is periodic and it may have some other pars which are not periodic and so on.

So, in order to get some feeling for this and just to exploit the fact that we already have these techniques and we will look a few different examples of this kind. So first thing I want to consider is what happens if $F(t)$ is a constant and if that constant happens to be a 0 that a case where it is an undriven problem. So we all know the answer to this undriven problem, so it is just going to give you oscillatory motion.

So you might thing the movement you turn on a constant external force you know may be there is going to be oscillations because of this you know the oscillatory nature of the problem but also this external force the constant external force. So maybe there will be some kind of you know motion which will cause your particle to keep on running away from where it started.

So let us see whether this kind of intuition is reasonable or is there something else which happens. So first thing as always, okay for simplicity I am saying that this particle is at rest and at the origin at time t equal to 0. So you could consider some other initial conditions and play with this and explore what happens. So as always the first step is so this is actually a, so this is the first example I am considering so let me call it *a*.

So first step is to non-dimensionalize pause the video here and figure your own way of non-dimensionalizing this problem. It is a fairly straight forward one but even so should do this. So first thing, so my way of non-dimensionalization is a following is find the ω scale is easiest for this kind of a problem which is just $\sqrt{k/m}$.

And then I therefore get the time scale which is just $1/\omega_0 = \sqrt{m/k}$. And then the acceleration scale is F_0/m, F_0 is an external force which gives me a natural force scale. And therefore I get a natural acceleration scale from which also I get the distance scale. I mean you can say that maybe you do not need the acceleration scale in this case you can directly to x scale it is okay.

We will put this down in a standard form that we have been looking at this problem. So once we have position and time scale, these are the two critical ones we make the transformation wherever you find x you replace it by the position scale times x, right. So now the new x is actually a dimensionless position in some sense, right. It seems like a something contradictory on the one hand we are calling it position but also saying it dimensionless.

So this is just a physicist way of saying that you are looking at a quantity which is really position but in such units so that you have basically scaled away these units. You have just making converting it into a purely non-dimensional quantity so that is what it is meant by dimensionless position there is also dimensionless time which is $\frac{1}{\omega_0}t$.

So then you get as usual so we will go ahead and substitute wherever we have x we put $\frac{F_0}{k}x$. So d^2x is just x, it is not should not get confused and think that all there is a second derivative so it must be x^2 it is not x^2 it is just only 1 x. But in the denominator there is going to be t^2.

So that is what gives me ω_0^2 so $m\frac{F_0}{k}\omega_0^2$ and d^2x/dt^2 with both of these quantities x and t are now in non-dimensional in nature is equal to $-k\frac{F_0}{k}x+F_0$. So there is external force which has to be included and so if you do take care of the algebra so and recall that $\omega_0^2=k/m$ lot of cancelations happen.

And then you get this very simple differential equation the second derivative $d^2x/dt^2=-x+1$. So we find that in fact the final differential equation after non-dimensionalization has no free parameter. It is just a simple differential equation involving no other parameters.

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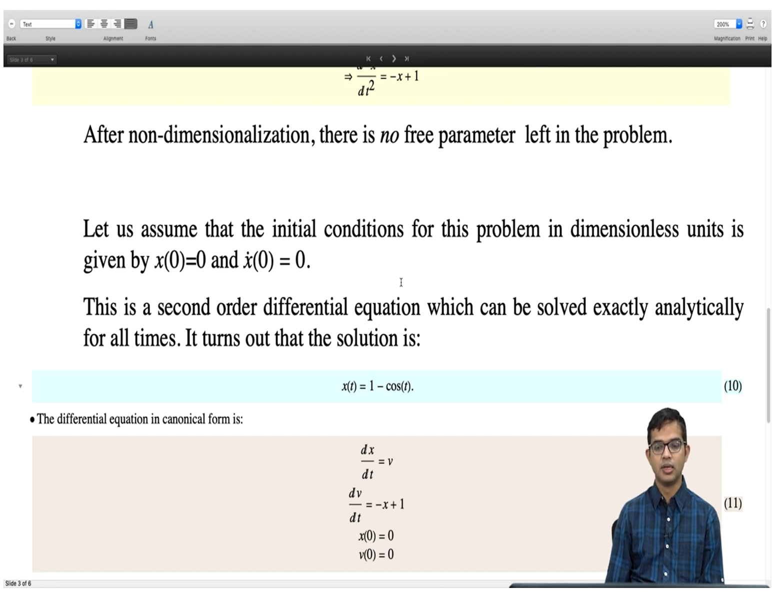


$$\frac{dx}{dt} = v$$
$$\frac{dv}{dt} = -x + 1 \tag{11}$$
$$x(0) = 0$$
$$v(0) = 0$$

- So in vector form we have:

$$X = \begin{pmatrix} t \\ x \\ v \end{pmatrix} \quad F = \begin{pmatrix} 1 \\ v \\ -x+1 \end{pmatrix} \tag{12}$$
$$\dot{X} = F$$

- So we proceed to define the functions and the initial vector:

```
rateFunc[{t_, x_, v_}] = {1, v, -x + 1};
initial = {0, 0, 0};
solx[t_] = -Cos[t] + 1;
```

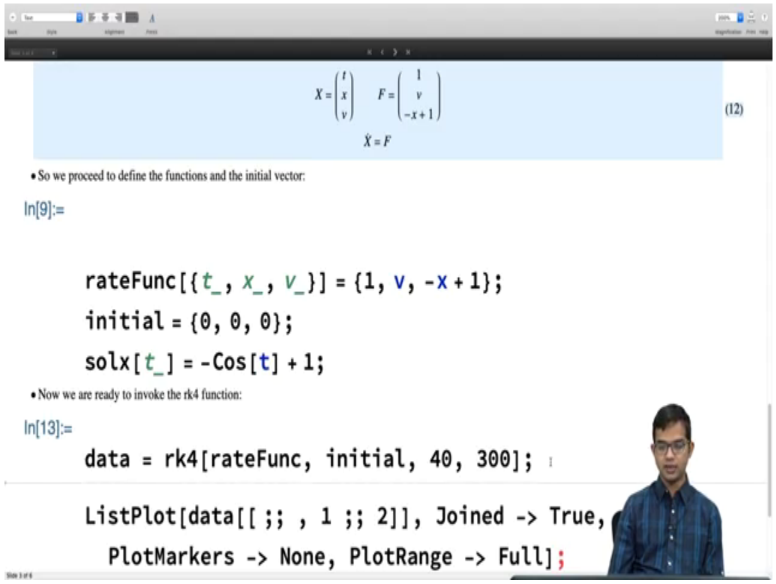
- Now we are ready to invoke the rk4 function:

$$R_2 = F\left(X_n + \frac{h}{2} R_1\right)$$
$$R_3 = F\left(X_n + \frac{h}{2} R_2\right) \tag{4}$$
$$R_4 = F(X_n + h R_3)$$

$$X_{n+1} = X_n + h \frac{R_1 + 2R_2 + 2R_3 + R_4}{6} \tag{5}$$

- Here we have copied its implementation.

In[2]:=
```
rk4[F_, X0_, tf_, nMax_] :=
 Module[{h, datalist, prev, rate1, rate2, rate3,
   rate4, next},
  h = (tf - X0[[1]]) / nMax // N;
  For[datalist = {X0},
   Length[datalist] ≤ nMax,
```



Okay, so we have already said that the initial conditions are taken to be $x(0)$ and $\dot{x}(0)$ is also equal to 0 and which is an acceptable non-trivial initial condition for this problem. So if you had the problem where there was no external force then this would not be an interesting initial condition to start. Why is that? Can you go back and check that and answer for yourself why this could be an interesting initial condition to consider if F_0 were 0? Answer this question yourself.

Now let us, let us move on so let us see what happens, so you have, so this is a second order differential equation how do we solve this differential equation? So there is a whole technology associated with solving problems of this kind, right. So it involves first of all rewriting it as some you know the left hand side must have only stuff which involves where an x is present, right.

So it is d^2x/dt^2 plus you can have some stuff constant times dx/dt plus some other constant times x equal to all stuff on the right hand side will be purely functions of t, right. So this is one way of separating it out and then you look at the homogenous differential equation solve as a, right down the general homogenous differential equation.

And then you need to find only one particular solution for the full in-homogenous equation and then if you just add one particular solution to the full general solution to the homogenous equation that in fact gives you the full solution for the inhomogenous equation. And then you have to just bring in the initial conditions and the two free constants which will come invariably for a second order differential equation.

We will all give fixed based on the initial conditions we have two initial conditions so two constants will be formed out and you get the answer. So for this problem and I am going to take this elaborate process I will just write down answers, I have wiped this out you can also check this yourself.

So it turns out that the solution is simply $x(t) = 1 - \cos(t)$. So it is a very simple solution and it is quite remarkable that you know in the presence of an external force. The particle is actually not going to not going to run away to infinity in some sense. It is going to still keep on oscillating about a different origin the only thing that this external forces manages to do is to just shift the origin about which this particle will oscillate.

So let us see what that means when we plot in a movement but before we plot let us use our very powerful tool box, the RK4 method, right. So in order to send our equation differential equation into this RK4 tool box you must first recast this second order differential equation into a, you know, more than one first order differential equation involving two variables.

As we have been doing to many problems we already know how to do this is take $dx/dt = v$ and then $dv/dt = -x + 1$. And then you have $x(0) = 0$, $v(0) = 0$. So in vector form we have you know this vector which also brings in time then x and v, x is equal to this vector and the rate of change of x the vector F is given by 1, it is always going to be the first row of this vector F is always 1 because $\dot{t} = 1$.

And then $\dot{x} = v$ by definition and then $\dot{v} = -x + 1$. So this is the problem we are looking to solve, right. So once again we will use the standard code so we have defined this rate function, rate function is a vector and it is a vector of functions if you wish.

And it involves this variables p, x and v which all have to be you know put in with a underscore sign because we are defining a function here and then. So let us go ahead and hit shift enter here so there we go, so initial vector is (0, 0, 0) time $t = 0$ the particles is at the origin and its speed is 0.

So it is (0, 0, 0) and the solution we already know is $-\cos(t) + 1$. So I will also define a new function which is called solution of t and now we are of course ready to go ahead and invoke the

RK4 function. So data is equal to RK4 of *ratefunc* comma initial comma 40 I am taking it up to 40 times steps. And 300 is a measure of the number of iteration involves.

So let us quickly go back and recall the arguments that go into this function. So n max so it is just the number if you make that larger than it is going to involve more operations and perhaps we will give better accuracy. But I am just choosing it to be 300 and I know that this is going to work well. So I am going to hit shift enter so it has already generated the data for me.

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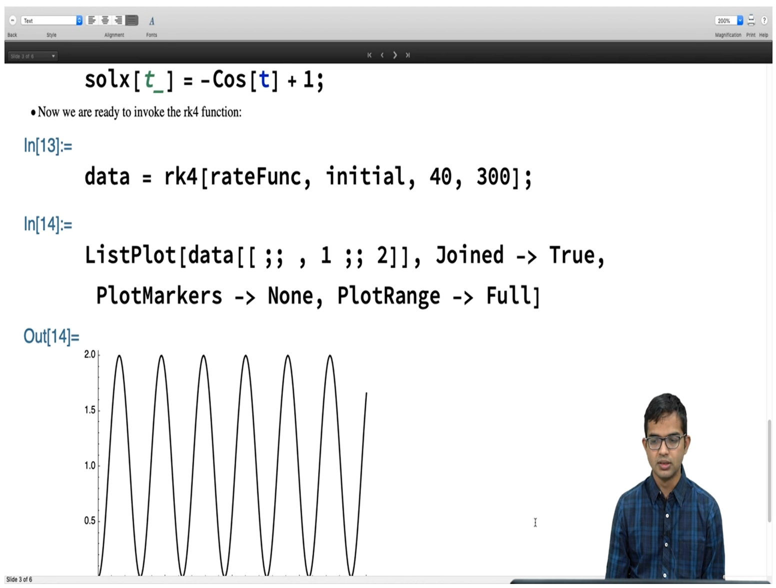
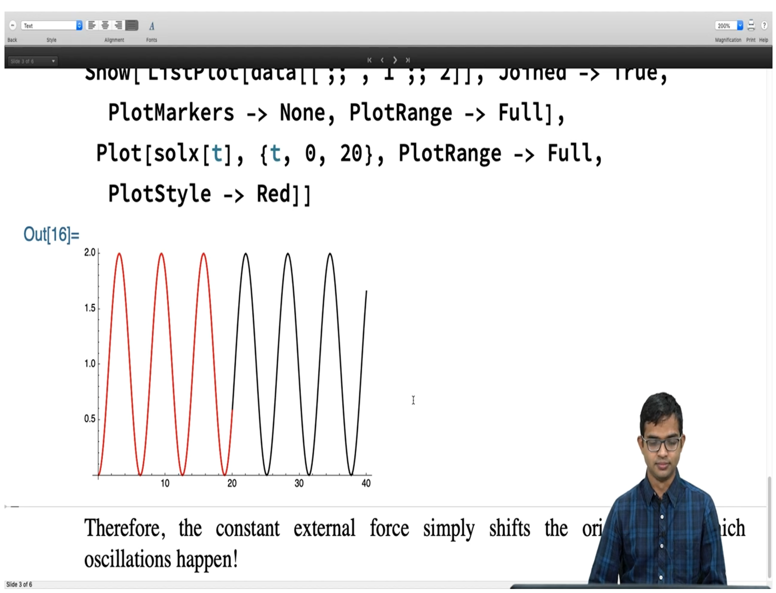




I can go ahead and do a list plot of this, there you go. So this is the solution coming out of the numerical implementation. So let me hide this and then compare against my analytical solution and there you go. So you see that it is perfect oscillations so the solution is so first of all there is perfect agreement between RK4 and my analytical solution.

And it is interesting that the analytical solution for this problem is also so simple, right. It is just an oscillation it is going to give you the same cosine t oscillation if you had $F_0 = 0$ then also you will get $\cos(t)$. I mean I am generally you can you could get $\cos(t), \sin(t)$.

So I told you to think about what will happen if you started with this initial condition $x(0) = 0$ and $\dot{x}(0) = 0$ with $F_0 = 0$.

So you will simply the particle will just remain at the origin forever so it is not very interesting. So probably you should choose some different initial conditions if you are working on the problem with $F_0 = 0$. And then you will get may be you can choose $\dot{x}(0)$ to be some 1 for example some other speed and then you will have oscillations which will be just cosine t plus sin t some linear combination of these two according to initial condition.

But here we see once again that you also get perfect oscillations there is no tweaking of this it is just oscillations and a particle is going to be bounded and does not matter how large your external forces it is not going to manage to shift the equilibrium position nothing else. So this is what something quite interesting.

So if you want your external force to do something more dramatic so you need it to change as a function of time. So saw that ofcourse if you had F_0 where itself a cosine function that is what problem we considered first.

We saw that it could if you add resonance it could give you like really large amplitude oscillations and it could take a particle I mean it is going to keep oscillating about the origin. But its amplitude can keep on increasing. So let us look at what happens if my external force is not periodic but it is constantly increasing as a function of time.

(Refer Slide Time: 18:34)

A linearly increasing external force

We are studying differential equations of the type:

$$m\frac{d^2x}{dt^2} + k\,x = F(t), \tag{13}$$

where $F(t)$ is a generic external force. Let us now consider an external force that is linear in time.

Exercise

(b) $F(t) = a\,t$.

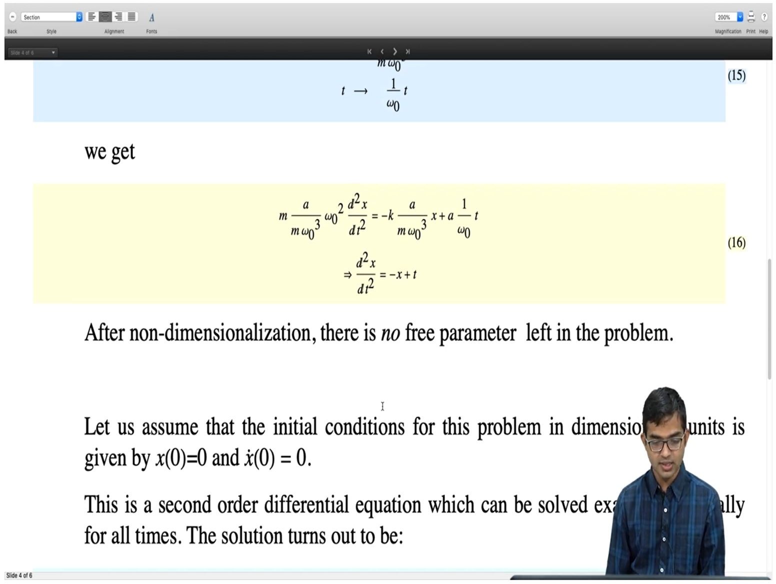
Solution

$$\begin{aligned} &\omega \text{ scale}: \quad \omega_0 = \sqrt{\frac{k}{m}} \\ &t \text{ scale}: \quad \frac{1}{\omega_0} = \sqrt{\frac{m}{k}} \\ &\text{acceleration scale}: \quad \frac{a\,t}{m} \\ &x \text{ scale}: \quad \frac{a\,t}{k} = \frac{a}{k}\sqrt{\frac{m}{k}} = \frac{a}{m\,\omega_0^3} \end{aligned} \tag{14}$$

Making the transformation:

$$\begin{aligned} x &\longrightarrow \frac{a}{m\,\omega_0^3}\,x \\ t &\longrightarrow \frac{1}{\omega_0}\,t \end{aligned} \tag{15}$$

we get



So the most natural function that comes of mind of this kind is a linear linearly increasing external force, so that is external force a linearly increasing external force. So we are studying this differential equation and now we take $F(t)$ to be some constant times t so I have chosen that constant to be a it is not be confused with acceleration a is just some variable. It is a parameter of your system that you can weigh.

So as always start by non-dimensionalizing your differential equation and you can pause the video here spend a few minutes and work out your non-dimensionalization and then cross check against my after a movement. So my solution is just the following as always I take my ω_0 to be just $\sqrt{k/m}$ therefore it gives me t which is $1/\omega_0$.

Acceleration scale in this case is at/m, right, so for t I will use I have to use this t scale. And therefore once I have I can get also x scale which is just at/k so you can see from this equation that force by k is just position. So I can use this at/k and then t is just nothing but $1/\omega_0$. So then with some small manipulation you can quickly convince yourself that the length scale in this problem is $\dfrac{a}{m\,\omega_0^3}$.

So if I plug-in again wherever I find x I must put length scale times x which $\frac{a}{m\,\omega_0^3}x$ and wherever I find t I must put $\frac{1}{\omega_0}t$. So we are doing this exercise in non-dimensionalization so often so much repetition of this because we have found that lot of that students are shaky on this.

Apparently it is simply thing to do but it throws a lot of students into confusion mode. So it is just trying to illustrate here that in fact it is very simple we just find all the scales in the problem and then you know using this technique not only you can do it quickly you can also now you do not have to introduce a new variables.

So that is also one way of doing it, so you can think of each of these scales and then call them by new label and then you introduce some x divided by you know the m scale is equal to a new label. But here the idea is you just retain the labels as they are x will remain x, t will remain t but you must interpret it right in the end.

So what do we, wherever we have x we will replace it by length scale times x so we have $m\frac{a}{m\,\omega_0^3}\omega_0^2$, ω_0^2 is this coming from $1/t^2$ which is sitting in the denominator.

So which is equal to $-k\frac{a}{m\,\omega_0^3}x + a\,\frac{1}{\omega_0}t$. A lot of cancelations will happen and we will be left with just $\frac{d^2x}{dt^2} = -x + t$. In the problem that we considered earlier it was $\frac{d^2x}{dt^2} = -x + 1$.

Now there is dependence on time and that is a linear dependence. So what would be expect, so now ofcourse we expect that since it is a force that is external force which keeps on increasing as a function of time. You do expect that your particle will not stay bounded it is going to be unbounded motion we will that in fact this it is true. But there must also an oscillatory component to this and so indeed that will also hold out.

So by the way there is no free parameter left and so we do not have any other you know knob to tune to change to some qualitative aspect of your solutions. So just back with this simple problem solve for it that gives you all the you know the most general behaviour of your system is contained in this problem involving no free parameter.

So we have already taken $x(0)$ to be 0 and $\dot{x}(0)$ to be 0 it is not an issue because this external force is anyway going to kick it out of its initial positions so there is no issue. It is just that if you did not have external force and if you took this form then it would be a dull problem and it will just lie there on the origin forever.

Now this is second order differential equation which can be solved analytically so once again one can invoke these techniques. So the solution for the homogenous equation the general solution is just always the same in these kinds of problems. It is just going to be some constant times $\cos(t)$ plus another constant times $\sin(t)$. So all the, you know variations are coming only from the type of external field which is being applied.

And therefore only the particular solution will change, and the particular solution you will see you will start to see a pattern. So in a slightly more advanced course on math methods or you know some differential equation course. In fact you will see that you know this whole class of differential equation there is very nice well worked out prescriptions for solving this. There is a way to get at the particular solutions.

So here I am going to once again just give you the answer it is a very simple answer.

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Let us assume that the initial conditions for this problem in dimensionless units is given by $x(0)=0$ and $\dot{x}(0) = 0$.

This is a second order differential equation which can be solved exactly analytically for all times. The solution turns out to be:

$$x(t) = t - \sin(t). \tag{17}$$

- The differential equation in canonical form is:

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -x + t \\ x(0) &= 0 \\ v(0) &= 0 \end{aligned} \tag{18}$$

- So in vector form we have:

$$X = \begin{pmatrix} t \\ x \\ v \end{pmatrix} \quad F = \begin{pmatrix} 1 \\ v \\ -x + t \end{pmatrix} \qquad \dot{X} = F \tag{19}$$

- So we proceed to define the functions and the initial vector:

Slide 4 of 6

```
rateFunc[{t_, x_, v_}] = {1, v, -x + t};
initial = {0, 0, 0};
solx[t_] = -Sin[t] + t;
```

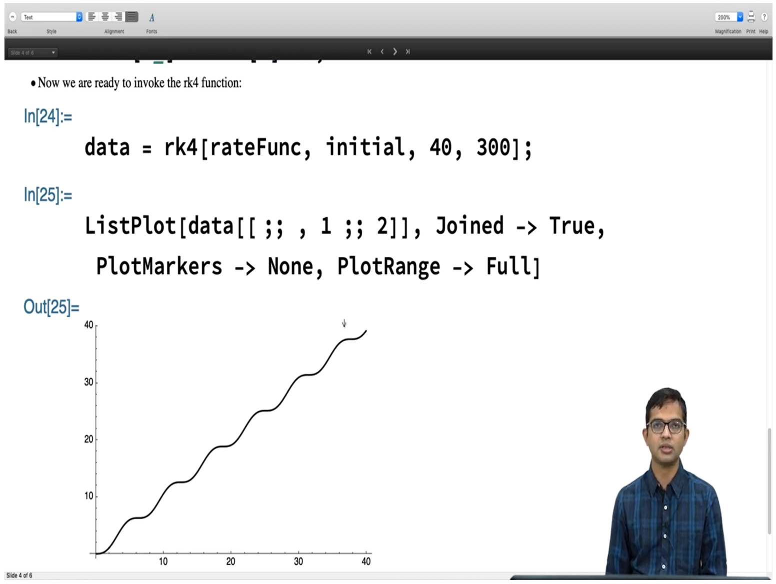
- Now we are ready to invoke the rk4 function:

```
data = rk4[rateFunc, initial, 40, 300];

ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True,
  PlotMarkers -> None, PlotRange -> Full];

Show[ ListPlot[data[[ ;; , 1 ;; 2]], Joined -> Tru
   PlotMarkers -> None, PlotRange -> Full],
  Plot[solx[t], {t, 0, 20}, PlotRange -> Full,
   PlotStyle -> Red]];
```

Slide 4 of 6



So $x(t) = t - \sin(t)$. So there is a $\sin(t)$ type of term, earlier we had $\cos(t)$ should be there either a $\cos(t)$ or $\sin(t)$. So in this case this turns out to be $\sin(t)$, we I mean it can be arbitrary combination of these two depending upon the initial condition. So it turns out that for this problem it is just $t - \sin(t)$.

And t is you know coming because of the external force, so as a function of time the position of your particle is going to keep on increasing. Not only is it going to keep on increasing but actually it is going to increase linearly with time. And then these oscillations only give you some fluctuations about this linear form.

So once again the canonical form is the $\frac{dx}{dt} = v \frac{dv}{dt} = -x + t$, $x(0) = 0$, $v(0) = 0$, all of this information can be encoded in vector form like here. And then we have vector x dot is equal to f rate function in this case is 1, v and $-x + t$ so I am going to hit shift enter initial conditions once again (0, 0, 0).

Solution now has changed, I told you the solution is $t - \sin(t)$, so I have the solution here. Once again we invoke the RK 4 I continue to use 40, 300 I mean we can tweak this and play with this we will see what happens, okay I do not need to make this verbose but this one I will show you what it looks like, so there you go. So it is a very interesting plot, so it shows you that it has the sort snaky behaviour.

It is like the system really wants to oscillate but then on average it also needs to keep on running away from the origin. So there is an external force which is kicking it, and it is going to keep on running away as a function of time. Yeah, so it is interesting that this external field which is proportional to time is going to give you also displacement which is roughly proportional to time only.

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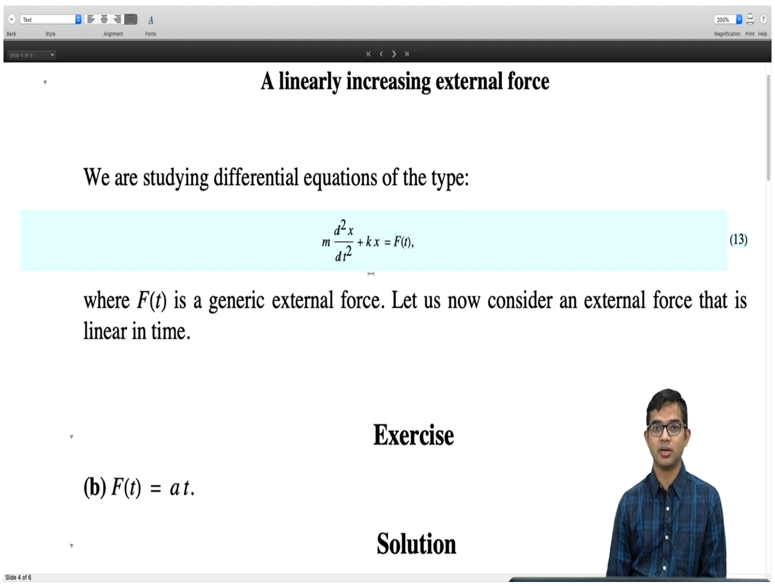

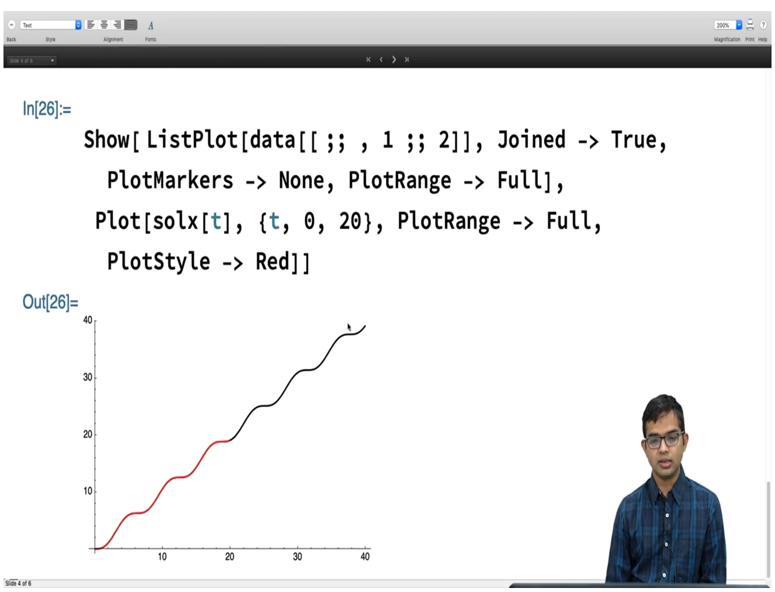

Suppose you do not have this dx/dt term, suppose we are we had a differential equation which only involved an external force and if there was no harmonic type potential in play. And if there were no k if there were no spring attached to this then you have to take you know integrate it twice. So it is going to be actually quadratic in nature.

So you see that there are two things that this the spring is doing. On a one hand it is trying to do bringing some oscillatory motion, but also it is also giving you it is also reducing your, the speed with which you run away to infinity. And it is in fact brought it down to linear.

And the final check is ofcourse against the numeric which we know and expect that are agree and indeed. So the two of them do agree as good appreciation as you would like. And I am just using 300 in this measure of number of operations.

You can play with this thing I am sure that it will work out even for much smaller number here as well. So, let us do a quick check of what happens if my external force were not linear but quadratic in nature?

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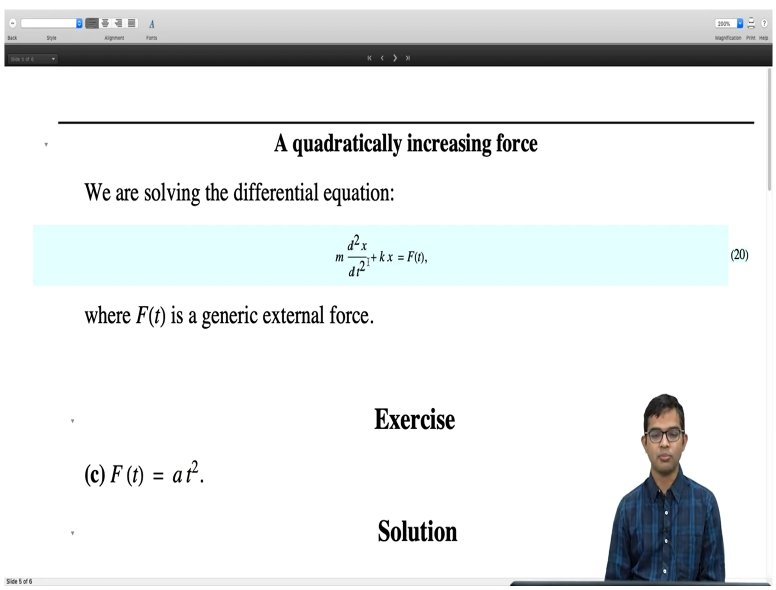


Solution

$$\omega \text{ scale}: \quad \omega_0 = \sqrt{\frac{k}{m}}$$
$$t \text{ scale}: \quad \frac{1}{\omega_0} = \sqrt{\frac{m}{k}}$$
$$\text{acceleration scale}: \quad \frac{a\,t^2}{m} \tag{21}$$
$$x \text{ scale}: \quad \frac{a\,t^2}{k} = \frac{a}{k}\frac{1}{\omega_0^2} = \frac{a}{m\,\omega_0^4}$$

Making the transformation:

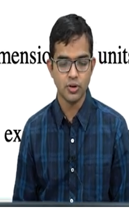
$$x \longrightarrow \frac{a}{m\,\omega_0^4}\,x$$
$$t \longrightarrow \frac{1}{\omega_0}\,t \tag{22}$$

$$t \longrightarrow \frac{1}{\omega_0}\,t \tag{22}$$

we get

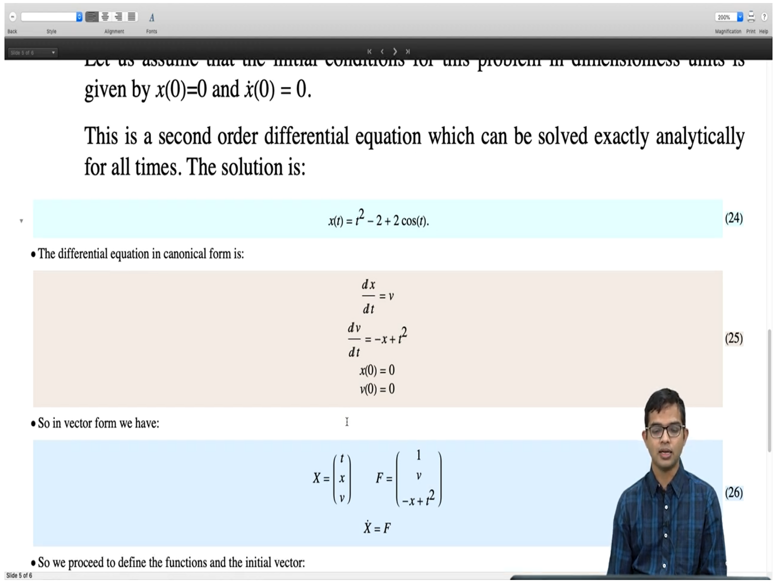
$$m\frac{a}{m\,\omega_0^4}\,\omega_0^2\,\frac{d^2x}{dt^2} = -k\,\frac{a}{m\,\omega_0^4}\,x + a\,\frac{1}{\omega_0^2}\,t^2$$
$$\Rightarrow \frac{d^2x}{dt^2} = -x + t^2 \tag{23}$$

After non-dimensionalization, there is *no* free parameter left in the problem.



Let us assume that the initial conditions for this problem in dimensio units is given by $x(0)=0$ and $\dot{x}(0) = 0$.

This is a second order differential equation which can be solved ex ally for all times. The solution is:



So what do you expect? You can pause the video for a moment and not do any calculation but ask yourself if you can guess how will this system behave for in the presence of a quadratically increasing force? So let us look at the quadratically increasing force case now. So I have $F(t) = at^2$. Now a once again is not acceleration a is a different constant different units and my solution for the non-dimentionalization is soon coming up.

So pause you video and work your own non-dimentionalization. Okay, ω is just the same $\sqrt{k/m}$, therefore time scale is $\sqrt{m/k}$, acceleration is scale at^2/m, t^2 I am going to use $1/\omega_0^2$, x scale is just F/k which is $\dfrac{at^2}{k}\dfrac{a}{k}\dfrac{1}{\omega_0^2}$.

So I will just write it as, a by m omega naught to the 4. Earlier I had a by m omega naught cube now it has just become a by m omega naught 4. And now you see that if I make the transformation x going to a by m omega naught to the 4 times the x and t going to 1 over omega naught times t. I will get a string of cancelations on both sides I urge you to verify this, right.

So you will see that on both sides lot of calculations will happen. And crucially what has changed now the right hand side is I have a by omega naught squared because there is a t squared sitting there.

And so lot of these things will just cancel out and finally my differential equation is a very simple $d^2x/dt^2 = -x + t^2$. So once again no free parameter, $x(0) = 0$, $\dot{x}(0) = 0$. If I impose these initial conditions I can ofcourse use a general theory of differential equations of this kind, you know linear differential equations in homogenous ones and I have a technique to solve for it.

But let me allow you to guess, what do you think will be different this time? So ofcourse we must have something $\cos(t)$ and something like $\sin(t)$ and whether you have only one of these two or both of these two will depend on $x(0)$ and $\dot{x}(0)$ and also ofcourse you know that precise form of the particular solution. So we saw that when I had just F_0 the particular solution was just a constant I had a one.

And when I had the particular solution brought me a t and now when I have an external force which is t^2, so what do you think it will be it is going to be quadratic in nature. So it is going to be in this case just t^2, so this is the solution. But in general actually the theory tells us that if you have t^2 sitting on the right side you could try out an ansatz like $\alpha\, t^2 + \beta\, t + \gamma$.

And then you have to find these coefficients. So there is a technique called the method of undetermined coefficients. And so it basically tells you what is the right type of form you must give for the particular solution, and then you plug that in and then you can actually extract those coefficients. And so then there is some more subtle aspects involved with this which is there is auxiliary quadratic equation that you can write for the homogenous part.

And then if one of the roots of this actually agrees with a certain coefficient on the right hand side, then you have to be a little more careful and so on. So but in general if you have a polynomial siting on the right hand side of a certain degree, the particular solution will also involve a polynomial of the same degree. So this is something that you might encounter in a full-fledged course on ordinary differential equations.

For our purposes I am telling you that the solution for this problem is just $t^2 - 2 + 2\cos(t)$. You can go ahead and plug this in back end to the differential equation and verify that it is true. So now, what should we do? We should recast this second order differential equation into the canonical form involving a first order differential equation of a vector of multiple functions.

So this x and v and I have $\dot{x}$ and $\dot{v}$ and then not only do I do this but I have to recast everything all of this put together including time into vector form. And then I have $\dot{X} = F$ where F is equal to 1, v and $-x + t^2$ that is what has changed.

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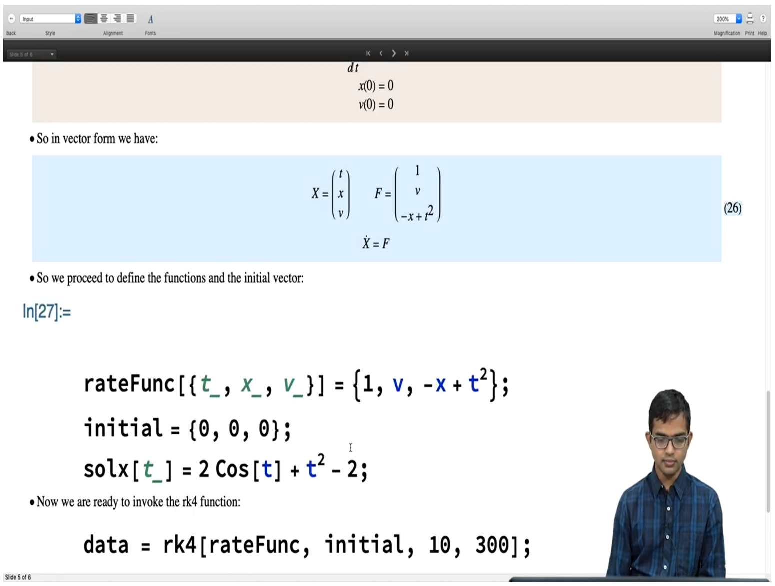

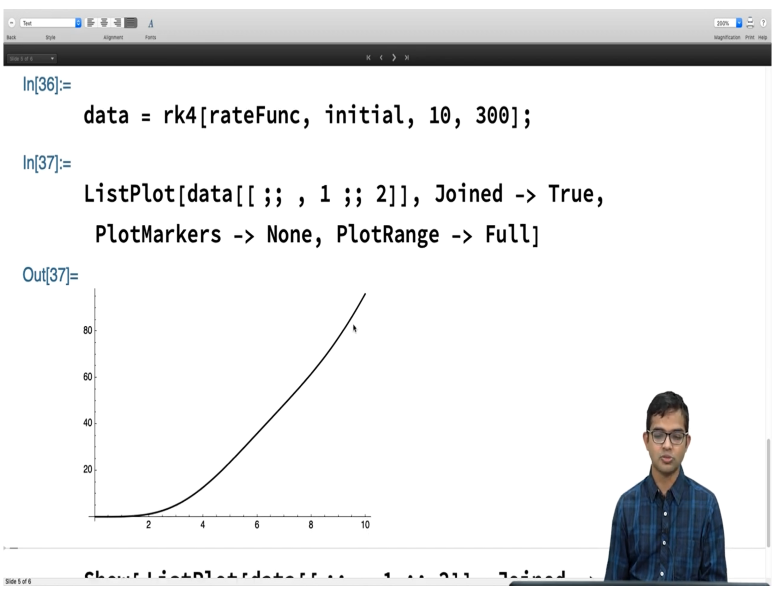

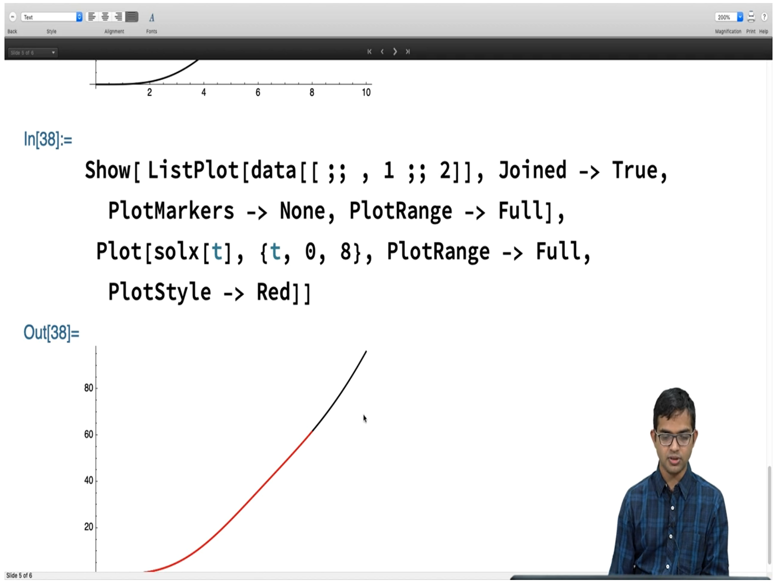

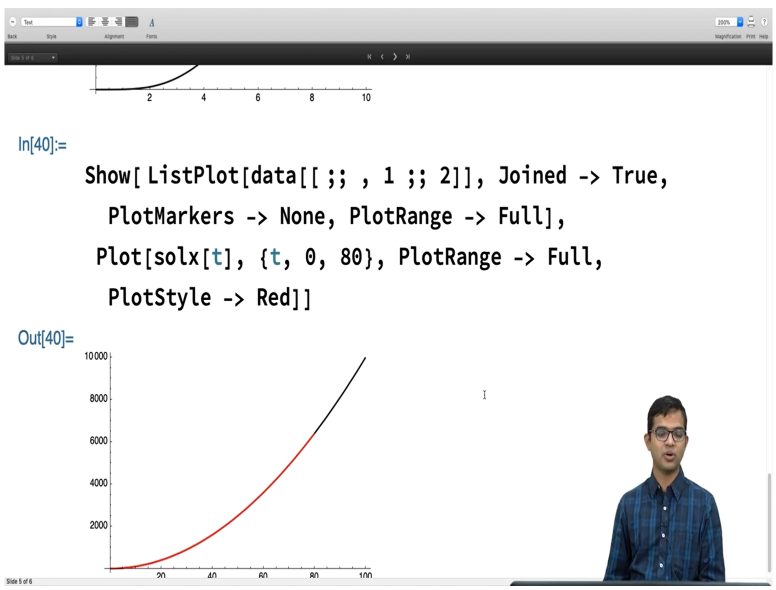

So then we have this rate function which also undergoes modification initial vector remains unchanged, initial vector is a (0, 0, 0). And then I have my solution has changed $2\cos(t) + t^2 - 2$ and then we go ahead and invoke the RK4 function. I am only going up to 10 this time. So you will see in a moment that.

Yeah, so this is what the solution looks like. So I mean it is basically quadratic in nature. I have chosen a small enough t so that I can see the small wiggles which you can see, right. Because t^2 is already a sufficiently fast growth that it is actually not going to it is not going to make this very visible. So this fluctuation, and then indeed there is also cosine in there, so as we can verify by cross checking my analytical expression with this.

So indeed the both of them agree very-very closely. So if I make this instead of 10 if I make this 100 so you will see that it is hardly going to be visible if I make it a 100, and then if I run this also make this like 80 let us say. And then you see that it basically looks like t^2, so it is you know very heavily dominated by the external force.

But once again I find it quite interesting that a second order, quadratically increasing force is giving you dynamics which is also quadrilateral increasing. Just simply because you have also this there is a springiness associated with the motion of your particle and that spring is not so successful in giving it any oscillatory motion here. But it is restricting its running away and it is keeping it at quadratic in time.

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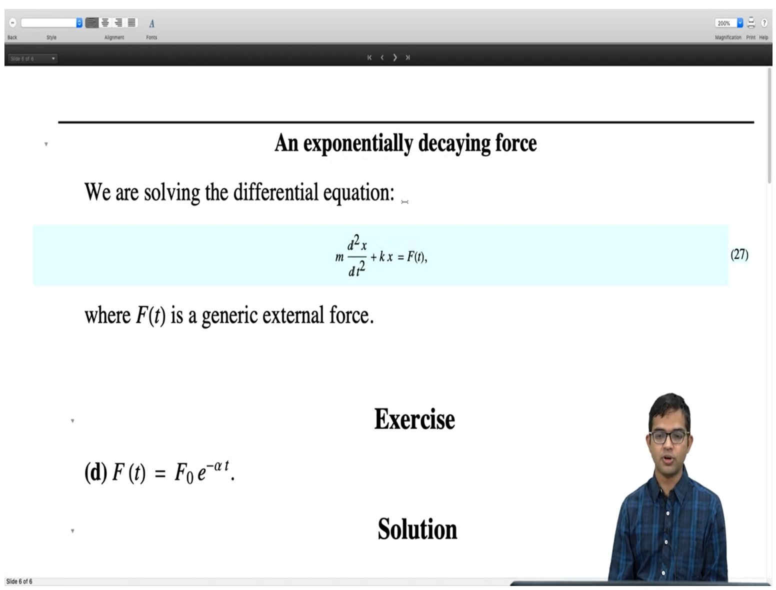


(d) $F(t) = F_0 e^{-\alpha t}$.

Solution

$$\omega \text{ scale}: \quad \omega_0 = \sqrt{\frac{k}{m}}$$
$$t \text{ scale}: \quad \frac{1}{\omega_0} = \sqrt{\frac{m}{k}}$$
$$\text{acceleration scale}: \quad \frac{F_0}{m}$$
$$x \text{ scale}: \quad \frac{F_0}{k} \tag{28}$$

Making the transformation:

$$x \longrightarrow \frac{F_0}{k} x$$
$$t \longrightarrow \frac{1}{\omega_0} t \tag{29}$$

Making the transformation:

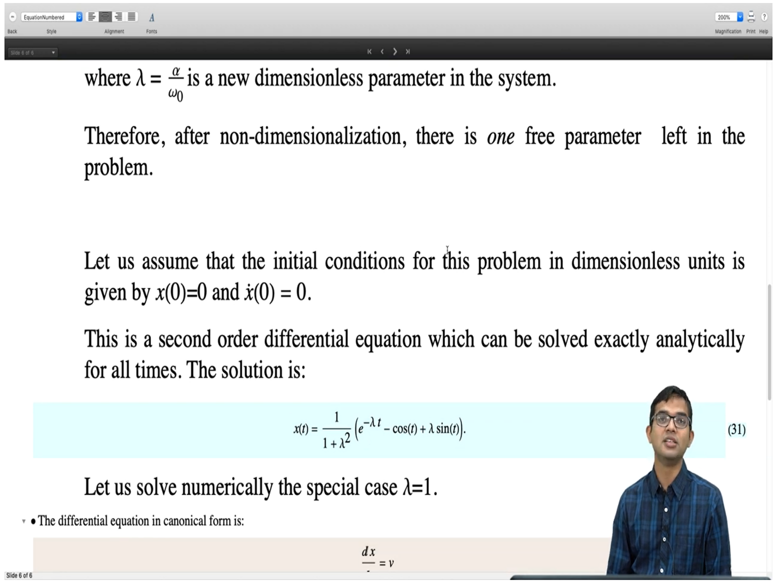
$$x \longrightarrow \frac{F_0}{k} x$$
$$t \longrightarrow \frac{1}{\omega_0} t \tag{29}$$

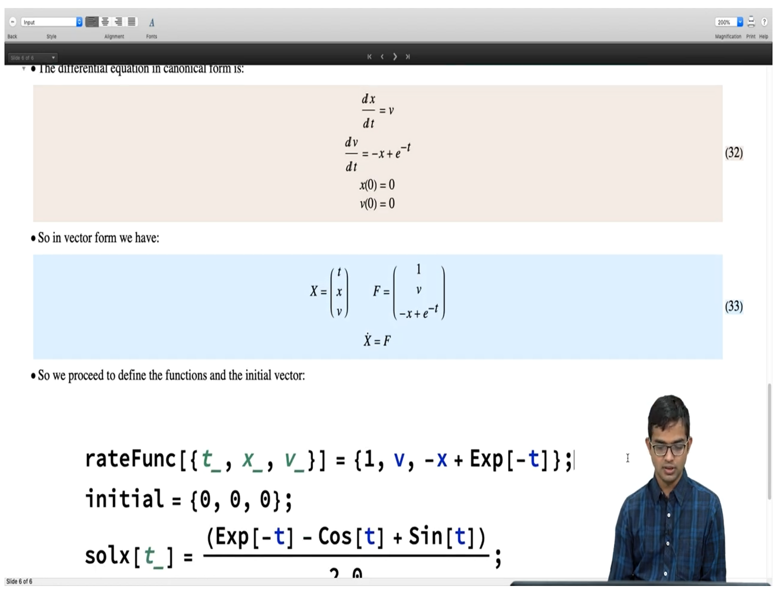
we get

$$m \frac{F_0}{k} \omega_0{}^2 \frac{d^2 x}{dt^2} = -k \frac{F_0}{k} x + F_0 e^{\frac{-\alpha}{\omega_0} t}$$
$$\Rightarrow \frac{d^2 x}{dt^2} = -x + e^{-\lambda t} \tag{30}$$

where $\lambda = \frac{\alpha}{\omega_0}$ is a new dimensionless parameter in the system.

Therefore, after non-dimensionalization, there is *one* free paramete… the problem.





So the final example that I want to do here is an exponential decay. So if your external force where started off with F_0 but if it were to decay down exponentially like here. I have another parameter α in here. Then that is an interesting external force to consider, so what kind of motion can you get? Will your particle be bounded, will you particle run away, what happens?

So let us check this, so I have to warn you that now the non -dimentionalization exercise here is a little less trivial than the previous one. So pause your video and see if you can come up with two independent non-dimentionalization schemes. I am going to use one but maybe you should come up with a different non-dimentionalization, solve your entire problem in a different non-dimentionalization scheme.

Then cross check that both the results should agree. So this is something that you should definitely do. So let me, so pause your video work this out and then you are ready to look at my solution. So my solution start simple $\omega_0 = \sqrt{k/m}$, t scale I will take it away $1/\omega_0$.

Now acceleration scale is $\frac{F_0}{m\,x}$, x scale is $\frac{F_0}{k}$. That was a very simple scheme I have chosen, so what could have changed? What could have changed is, we could have taken a different time scale for this problem. So alpha brings it an alternate time scale, instead of using $1/\omega_0$ I could have also used $1/\alpha$ as a time scale.

Or I could have used some linear combination of these two or something like 1 over square root of omega naught square plus alpha square for example or something else some that is a very complicated things.

So there are actually infinitely many ways of doing non-dimentionalization and depending upon the precise details of your problem may be one is more convenient than the other but now of these can be wrong. So I urge you to play with this, try out at least one more and may be many more non-dimensionalization schemes, extract the solutions and make them all agree with each other.

So in the spirit of this course is to basically solve simple problems but look at them from multiple angles and reinforce our understanding. So, make it so solid with simple problems that then automatically new fresh directions emerge and then suddenly we find ourselves solving new problems unsolved problems or problems which may be which may lie in the realm of research suddenly opens up.

So this is how research also those just that you become so strong with what is already know, then suddenly you keep on knocking on doors and some door opens up which actually leads you to something new and sometimes not just new but something very exciting can also come up in this same process. So that is what this course is about.

Now making this transformation I will change x to $\frac{F_0}{k}x$, t to $\frac{1}{\omega_0}t$ and then we get the string of simplifications. And now there is a small variation which appears. We have $\frac{d^2x}{dt^2} = -x + e^{-\lambda t}$. So it is no longer a parameter less final differential equation, so there is a new dimensionless parameter.

So you can see that I have introduce a λ which is actually α/ω_0. So it has the same so α and ω_0 have the same dimensions, so α/ω_0 is dimensionless. So $\alpha\, t$ is dimensionless and $\omega_0\, t$ is also dimensionless. So α/ω_0 is indeed a dimensionless parameter. So there is this one free parameter left in this problem it is still a completely analytically solvable differential equation.

So I have for simplicity I am just taking $x(0)$ to be 0, $\dot{x}(0)$ to be 0 and the solution it turns out I am just giving you the solution. You would expect that the solution will be a linear combination of $\cos(t)$ and $\sin(t)$. And then because there is a $e^{-\lambda t}$ siting on the right hand side, the theory tells us that we must try out for a particular solutions something which involves $e^{-\lambda t}$ times some constant.

So you can take care of all the details and work out. There is a very systematic way, it is not magic that I am able to write down the solution. There is a very well-known and well-delineated systematic approach for doing this. If you do it then you can show that $x(t)$ is given by this expression.

For our purposes it is suffices, if we can just plug this expression in back in to the original differential equation and check that indeed it holds up. Now, for the numeric so you should workout this problem for various different values of lambda.

But I want to choose lambda equal to 1 but you can maybe use some you know like a manipulate command and you know try out our range of λ's and see whether the nature of the solution changes substantially.

So by the way, the case of $\lambda = 0$ is interesting. So what does it mean? If I put $\lambda = 0$ and my original equation, λ is a same as α basically in units of this ω_0. So if λ is 0 that is basically α is 0 and that means basically $F(t) = F_0$ and is this a problem we already know how to solve?

Yes, we just did it in this very module. And then you expect the solution to be just $1 - \cos(t)$and that is what we are recovering the solution, so that is good. And the other extreme is $\lambda = \infty$. So in other words, if $\alpha \to \infty$ you see that $F(t) = 0$, right that is also a case that is familiar.

What is interesting is that, both these limits actually give you just perfect oscillations. I mean you would not recover that if you directly plug in here for the reason I told you earlier which is a we have chosen this special initial conditions which make the that problem where if you put $F_0 = 0$. If you put $F(t)$ is just 0 if you may then it is going to be an uninteresting solution. I just going to give you $x(t) = 0$.

But if you try out, so this is some is an exercise for you guys. I want you to check the limit of lambda going to infinity and see that it will reduce to a more interesting solution for the unforced problem by taking $x(0)$ to be some, $x(0)$ can be 0 but let us say $\dot{x}(0)$ is taken to have some finite velocity, may be 1 let us say or in some, some number you can choose.

And then you will see that, work out the solution and then take that limit and then you will see that indeed there is going to be a connection to the original cosine plus sine, $\cos(t) + \sin(t)$. So it is quite nice that an exponential decay function whether it is, so there are two limits to these.

So constant or F_0 and both of these limits will give us just pure oscillations. So now what happens if $\lambda = 1$, let us look at this. So we have indeed there is this exponential part which is actually like your transients so it is like does not matter what λ is. The system is eventually going to oscillate, it is not going to run away to infinity, because the force is simply not strong enough.

So we know that, you know, the two extreme kinds of this type of force are either it is constant force here the constant force only manage to give us oscillations but about a different equilibrium position. And the other extreme is just no force and that is the familiar problem where ofcourse you are not going to have anything other than oscillations about the mean.

So if you put an exponential decay in time also you expect that this motion is going to be bounded. So this is an argument you could have perhaps come up with right at the beginning itself. So now you will see that now that we have the solutions and we will explicitly check this numerical $\frac{dx}{dt} = v \frac{dv}{dt} = -x + e^{-t}$. And then we recast it in the canonical form, in the vector form like here.

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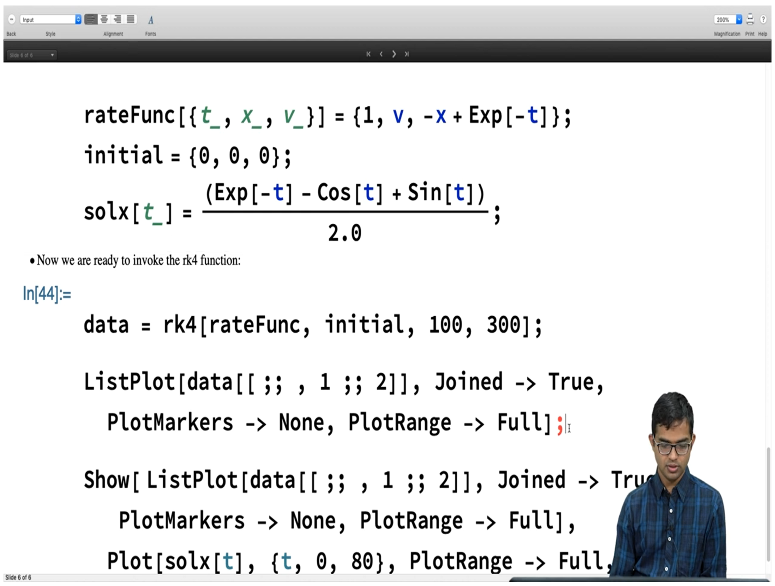
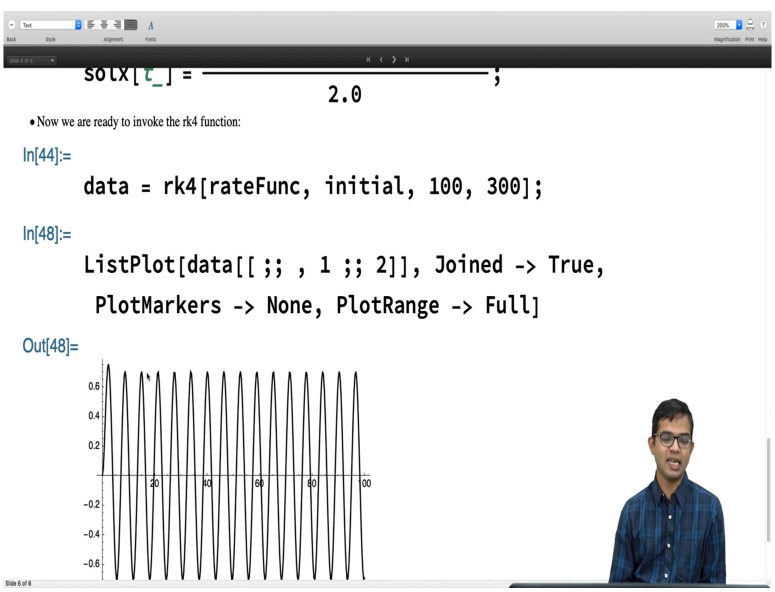




```
solx[t_] = ——————————————— ;
                 2.0
```

- Now we are ready to invoke the rk4 function:

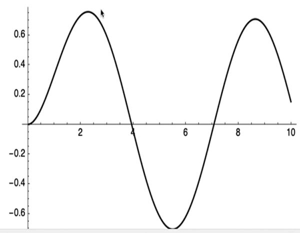
In[49]:=

```
data = rk4[rateFunc, initial, 10, 300];
```

In[50]:=

```
ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True,
 PlotMarkers -> None, PlotRange -> Full]
```

Out[50]=



```
solx[t_] = ——————————————— ;
                 2.0
```

- Now we are ready to invoke the rk4 function:

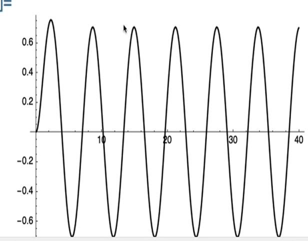
In[51]:=

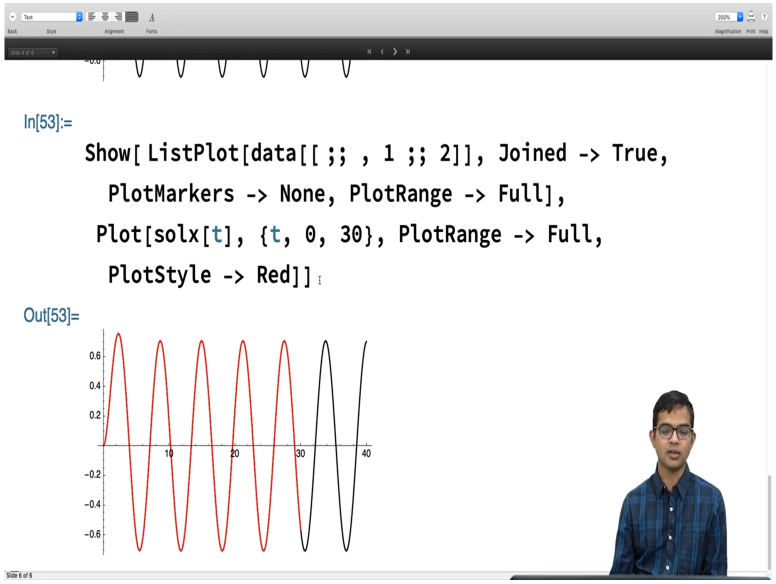
```
data = rk4[rateFunc, initial, 40, 300];
```

In[52]:=

```
ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True,
 PlotMarkers -> None, PlotRange -> Full]
```

Out[52]=





And then we have the rate function initial vector and then the solution all defined like here then we go ahead invoke the RK4 routine. Let us see what the solution looks like, so the solution looks like here. So indeed it starts of, you know, trying to do this exponential decay but this a very small effect it is going to be there only for a short time.

So instead of a 100 if I have made it just 10 then perhaps it will be more exaggerated this thing. So there you go initially it seems like it is going to keep on digging but it is not going to fall below a certain values because this $\cos(t)+\sin(t)$ will kick in for large t and e^{-t} is basically become irrelevant as times become large.

So if I take something like a 40, 40 time steps there you go you see that now it is basically its amplitude is a constant for all practical purposes and now if I had to go here and compare this against my analytical solution I would not be surprised that the two of them agree. And in fact that only adds to our confidence.

So we what have we done in this module we have played some games involving external forces of a very simple kind but which were not oscillatory in nature and manage to extract some very simple and very instructive physics out of this. And we manage to use RK4 to cross check our results. So by the way just before I close off I want to tell you that these are problem which appear in the very famed Landau-Lifshitz series.

So there is a very nice book by Landau-Lifshitz on mechanics but it is rather intimidating textbook if you were to just directly go and look into this section of oscillations you would find these problems. And for a person beginning out in physics Landau and Lifshitz would be a formidable place to learn it from.

And so I am pretty sure that somebody if you know if there are picking up, the first mechanics textbook that they picked up was Landau and Lifshitz it would be quite difficult for them to open up and understand all of these in a such a transparent way.

So what I have managed to show you is that you know problems of the level of the Landau and Lifshitz also now that we have worked it out in this explicit manner everything seems so crystal clear. And so in fact I urge you to go back and try out more problems from Landau and Lifshitz and even more challenging textbooks.

So now that we have this powerful technique you do not have to you know suffer through some very difficult algebra I mean you should do that as well. But in, addition you have this alternate approach cross check and then once you have understood it, it becomes easier actually to master the analytical tools. So on that note I sign off. Thank you.