Physics through Computational Thinking
Professor Dr. Auditya Sharma
Dr. Ambar Jain
Department of Physics
Indian Institute of Science Education and Research, Bhopal
Lecture 32
Falling Bodies

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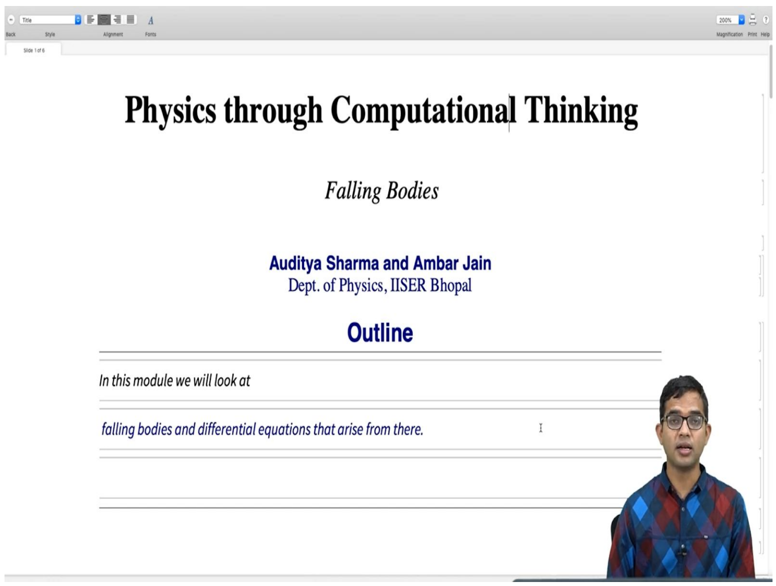

Okay, hello, so in this module we are going to look at the Physics of Falling Bodies, right. So this is stuff, some of the stuff that we will discuss here is familiar territory we have seen it very back in high school and but we will look at from somewhat technical point of view. We will start with thinking of certain differential equations which come out and then write it in a very solid way.

So that it is amenable to generalization and then we will also consider certain types of forces which perhaps you have not seen. And then of course because this is a computational thinking course we will see how we can break apart the differential equation and put it into a form which is amenable to feeding it into our computer and then extract numerical results and then come back and analyze.

So there are some nice techniques which, of differential equations which we will learn but also make use of routines which we have already discussed perhaps in some other earlier videos or we will make videos for these programs. So we will use certain programs and then cross check our results.

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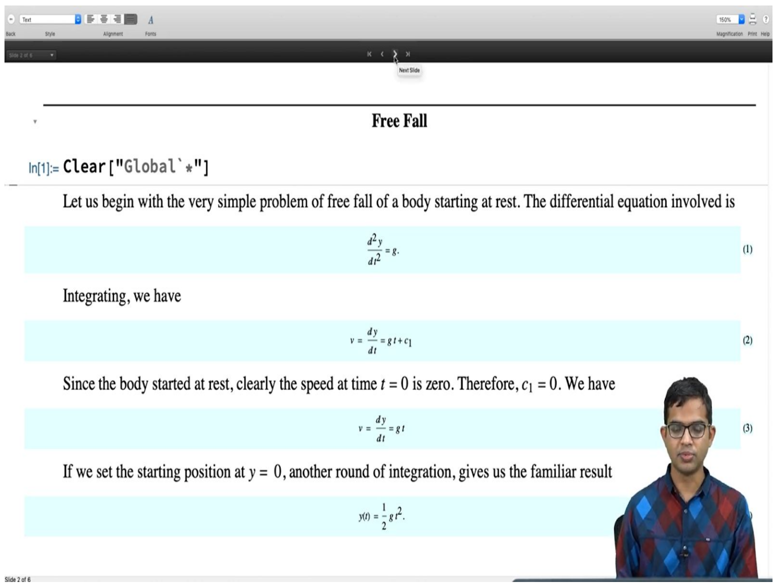

Alright, so as always, when we are starting with a Mathematica session it is useful to have this clear all options. So this syntax very important one has to be careful with the use of the right symbols in there and then you go ahead and shift enter and then your session has begun. It is just a way to avoid you know like overlapping variables and so on.

If you are starting a Mathematica session completely afresh when it really not required but sometime what happens is you have an old session which is running and some of the variables are already in there and then if you do not do this then there can be some conflict of variable.

So it is as a safety measure and as good programming practice it is good to start pretty much any session of Mathematica with this comment. So the problem we want to consider is that of a freely falling body and the motion that it goes through. So it is a very simple problem we know that it is a constant acceleration and the acceleration is purely due to gravity.

So in fact the correct differential equation to write here would be $m\,a = F$. So it should be $m\,d^2y/dt^2 = m\,g$. But when the mass cancels on both sides and you are left with just $d^2y/dt^2 = g$ which is really a kinematic equation.

So this is how we have learnt about Newton's law of motion and then we assume that you know if you are sufficiently close to the surface of the earth its gravitation due to the earth it is going to give you uniform acceleration and that is given by g. And then of course it is straightforward to solve this differential equation so you just integrate once and then you get the velocity as a function of time it is just given by gt plus some constant.

So here let us assume that you started at rest and you know and for simplicity, we can also take the starting point to be at 0 in terms of y. You put your y axis origin to be at where the body is when it starts at rest at time $t = 0$. So we have $v = g\,t$ because $c1 = 0$ and then you integrate once again.

And then you get ½ $g\,t^2$. So this is high school physics, very familiar territory so far. Now here what we want to do here is ask what happens if this free fall where all when the presence of some drag forces. So the most common kind of drag forces that we encounter here is the following.

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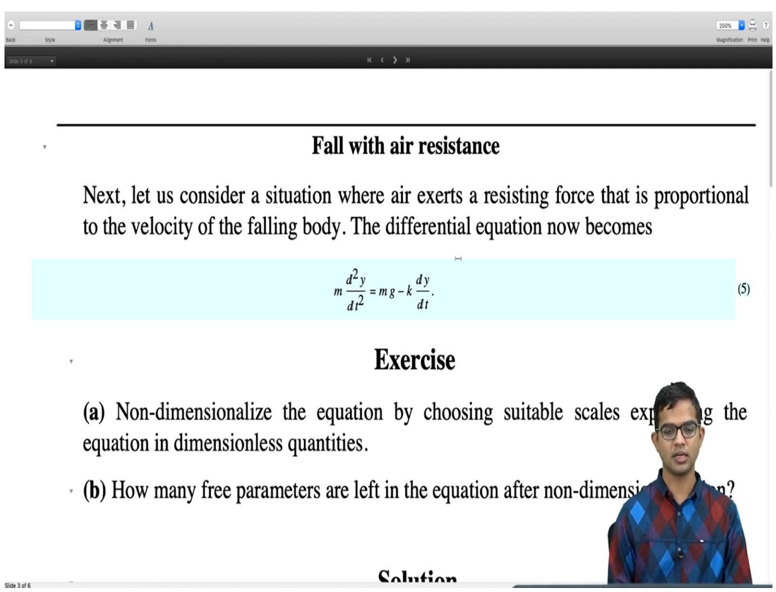

So let us look at a drag force which is proportional to the velocity. So the differential equation is going to be $m\, d^2y / dt^2 = m\, g$ minus there is a drag coefficient you can call it which is just which gives you the term $-k\, dy/dt$.

So the sign is important k is taken to be positive, so the more the speed with which the particle is falling down the more is the force that is trying to slow it down. And not only is it trying to slow down not only is it greater in magnitude but it is also in the direction opposite to the direction in which motion is happening.

So this minus sign is crucial. So you are given a differential equation like this the first step is of course to non-dimensionalize this. So if you want you should pause the video at this point and go back and carry out this exercise yourself. So we have seen already multiple examples of how to do this non-dimensionalization and it is very important exercise. So that is why we keep re-emphasizing this again and again.

So please go back and try this and then you can look at my solution. And so you should also address the question of how many free parameters are left. So apparently it looks like there lot of parameters there is mass, there is gravitational constant, there is another coefficient k. So are all of these going to factor into the essential physics of it is the qualitative nature of the solution going to dependent on all of these or how many parameters come out. So that is the one of the first important questions to address so my solution is going to follow.

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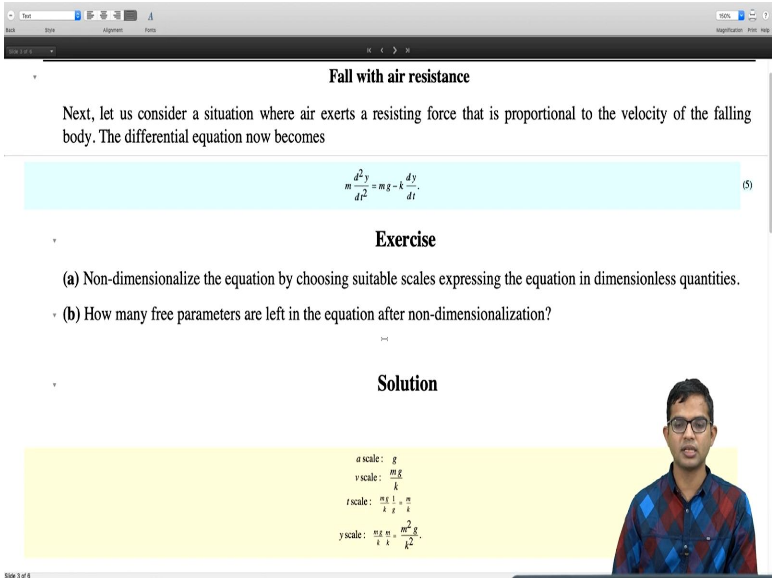

So it is easy to check for the acceleration scale, so the acceleration scale is simply given by g we already know this. Then the velocity scale comes by looking at the 2 terms on the right hand side. So there is mg must have the same units as $k\, dv/dt$, dv/dt definitely has units of velocity. So $(m\, g)/k$ better have you need some velocities.

So I can take these 3 constant $(m\, g)/k$ and form a velocity scale or a speed scale. And from which I can extract a time scale so speed divided by acceleration must give a time scale. So I have $(m\, g)/k$ divided g which gives just m/k I could get this perhaps some by different approach. But m/k is definitely a times scale.

So once I have a time scale with the help of the speed scale I can just multiply speed times time is a distance. So we can put $m\, g/k * m/k = m^2 g/k^2$. So this is my distance scale.

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$$\text{y scale}: \frac{mg}{k}\frac{m}{k} = \frac{m^2 g}{k^2}.$$

Making the transformation:

$$y \to \frac{m^2 g}{k^2} y$$
$$t \to \frac{m}{k} t \qquad (7)$$

we get

$$m \frac{m^2 g}{k^2 \left(\frac{m}{k}\right)^2} \frac{d^2 y}{dt^2} = m g - k \frac{m^2 g}{k^2 \frac{m}{k}} \frac{dy}{dt}.$$
$$\Rightarrow \frac{d^2 y}{dt^2} = 1 - \frac{dy}{dt} \qquad (8)$$

After non-dimensionalization, there is *no* free parameter left in the problem!

Let us assume that the initial conditions for this problem in dimensionless units is given by $y(0) = 0$ and […] 0.

This is a second order differential equation which can be solved exactly. The method involv[…] really a first-order differential equation in the velocity. Defining $v = \frac{dy}{dt}$, we have

Let us assume that the initial conditions for this problem in dimensionless units is given by $y(0) = 0$ and $\dot{y}(0) = 0$.

This is a second order differential equation which can be solved exactly. The method involves realizing that it is really a first-order differential equation in the velocity. Defining $v = \frac{dy}{dt}$, we have

$$\frac{dv}{dt} = 1 - v. \qquad (9)$$

This equation can be solved by the method of separation of variables:

$$\frac{dv}{1 - v} = dt. \qquad (10)$$

Integrating we have

$$-\log(1 - v) = t + c. \qquad (11)$$

Since $v(0) = 0$, we get $c = 0$. Therefore, we have

$$v = 1 - e^{-t}. \qquad (12)$$

A plot of this function is very instructive.

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In[2]:= vfunc[t_] = 1 - e^-t;
Plot[vfunc[t], {t, 0, 4}, PlotRange → Automatic, AxesLabel → {t, v
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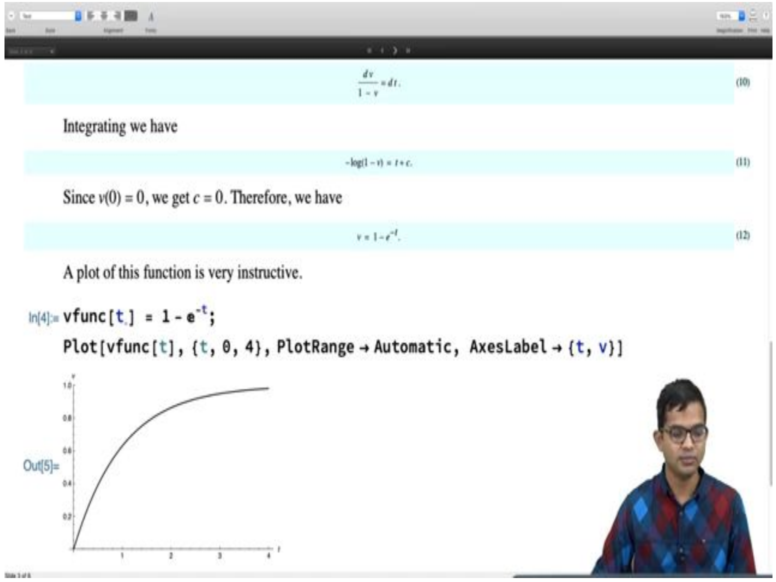

So now you know there are various ways go going about doing this non-dimensionalization one quick way is to just take any you know take your original equation and wherever you find position wherever you find y you replace it by the position scale times *y*. So you do not put only *y* but you put $m^2 g/k^2 * y$.

Wherever you find *y* and wherever you find *t* you replace it by $m/k * t$ so then what happens. I have *m* times in the numerator I must put $m^2 g/k^2$ because it is only *y* which appears here. But in the denominator there is t^2. So I am going to put $(m/k)^2$ in the denominator then I just leave d^2/dt^2 as it is.

And on the right hand side *m g* remains *m g* then I have $- k *$ in place of dy/dt I must write, I have to substitute for *y* and for *t*. So in place of *dy* I will write $m^2 g/k^2 \ dy$ and in the denominator in place of *dt* I will write it as $m/k * dt$.

Now these *k*'s cancel so if you are careful with algebra here you will find that you will get $m\,g\,d^2/dt^2 = m\,g - dy/dt$. So *m g* cancels throughout and then you are left with just the simple differential equation which is in the non dimensionalized form. And where as you can see there is no free parameter left in the problem.

So what it tells you is that really all the scales have been already absorbed and there is no essential dependence on any of the parameters any of the set of parameters as far as the qualitative nature of the physic is concern. So it is just that no matter what values you choose for $m\ g\ k$ in the problem, the essential it is only going to stretch your scale in some way it is going to not affect the qualitative nature of the differential equation.

So this is already an important message to take home, right from the problems you started with a differential equation where we input some physics. And then we have brought it to a concentrated form now which is purely a mathematical differential equation at this point in some sense we have removed all the physics from it in some sense at this time.

And now it is in a such concentrated form it can be fed into a computer but in this case it turns out that even before I feed it to the computer as a pure differential equation I can actually go ahead and just solve it. There is a way to say this problem. So for simplicity I am going to assume that I have already said this in this in the previous version.

I will continue with the same assumption that at time $t = 0$ the problem the particle in question is at the position 0. And it has a speed 0 and does not matter in which units. So it is 0 is 0. Anyway even in non-dimensional units is where I am supposed to be working is this going to be 0 and so this is a second order differential equation.

So, there are very general techniques how to about solving second order differential equation which are linear. This is not only second order but it is also linear. So it is not a surprise that an exact solution exist. But there is a nice way to reduce this second order problem to a first order problem.

It is not even as hard as solving a general second order differential equation which is linear. We have looked at an example of that type when we looked at the problem of damped harmonic oscillator and so on. So this is an even simpler problem so what you do is simply introduce another variable $v = dy/dt$ namely the speed.

And then you write it in terms of a differential equation of v involving v, so we have $dv/dt = 1 - v$ therefore an immediate separation of variables becomes possible. Then you have $dv/1 - v = dt$ and then it is just $log(1 - v)$ with an overall minus sign on the left hand side is equal to t plus constant. So now I have to invoke the initial conditions so I know that at time $t = 0$, v must be 0.

So the left hand side is also going to 0 and therefore the constant is 0. So the final solution as far as the speed of this particle is concern as a function of time is $1 - e^{-t}$. So contrast this with the previous problem where we had the speed was growing as a function of time, right.

Here you see that it is going to become really a constant for long times as t becomes larger and larger we are going to get a velocity which is going to saturate, right. And so of course a plot of this function is instructive so I will define this function I have told you the syntax for this you must use this underscore when you are defining this function, this some mathematica syntax.

And then I will just plot this, so there is a plot function plot range automatic and axis label are stated as like is appropriate and there you go. So if plot this as a function of time I am just looking at time is going from 0 to 4. So see that there is clear saturating tendency if you keep on extending it this to larger and larger values of t.

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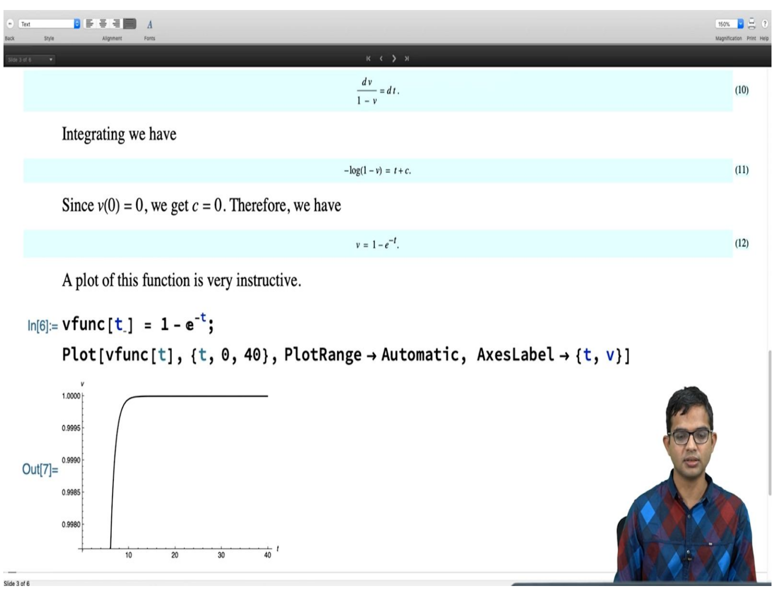

Let us see what happens when I make it 40 then it is going to saturate to 1 not a surprise because this e^{-t} is going to become smaller and smaller and it can never cross 1 so it will just tend to 1 discrete function. Okay, so one more integration is all it takes before we in fact have the solution for the distance covered by this particle as a function of time.

So this is a really this is free fall but it is, free fall in which there is also a resistance. So its speed is bond by certain value. So in this case $v = 1$, what is $v = 1$ mean?

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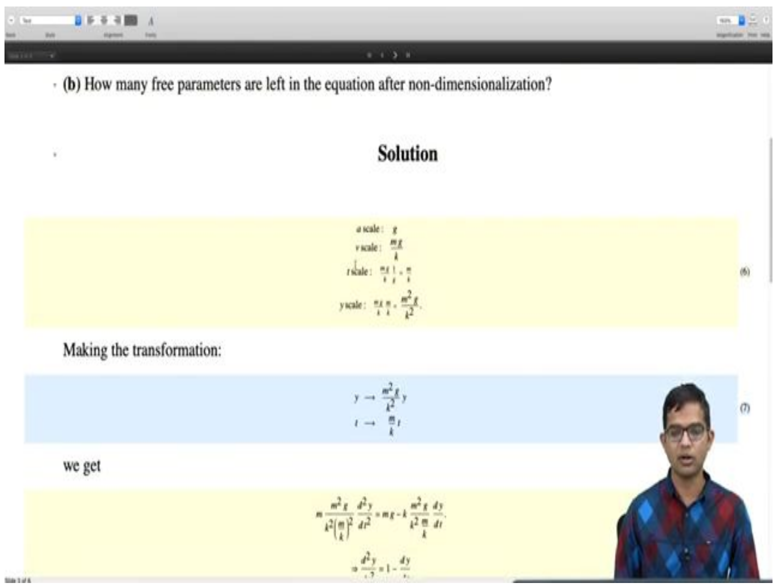

We will let us go back and look our, the scale of our velocity scale is $m\,g/k$. So basically what it tells you is that no matter where you started I mean this would be true also for other initial conditions you can play with other initial conditions and check this out.

So you will see that there always going to be a saturating tendency for this kind of a problem so you will have some if there is a differential initial condition you will get a different c here. And so that also going to still give you, you know 1 minus some constant times e^{-t}.

So that part will remain same for long times it is going to saturate to may be somewhat so slightly different value depending upon the initial speed. But ultimately, it will saturate and that the scale of that problem is set by $m\,g/k$. So in this current problem with the initial conditions we have chosen it is going to saturate exactly to $m\,g/k$. So that is how we have to interpret this.

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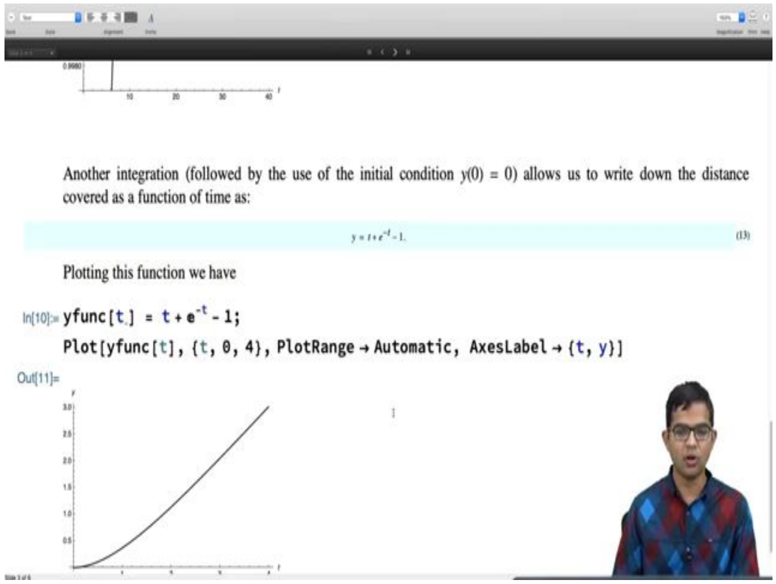

So another integration we can do then we have $t + e^{-t} - 1$. Where we have already incorporated to the initial condition that at time $t = 0$ the particle was at the position 0. So I can quickly check this if I put $t = 0$ here I have 1 + 1 0 + 1 - 1 which is your 0.

So indeed that part is good and if I take a time derivative of this I will recover this speed. So it is all good indeed this is the correct solution, so let me also plot this and so then we have we have this function there you go. And so you see that for long times this e^{-t} becomes less and less important because it is hardly changing.

And so all that count is just the t it is going to become a linear function for long time. So that is the essential message from this is that it is going to linear limit time for long times and that is consistent with the finding that the velocity is going to saturate. It is constant velocity problem. So this is all good we have managed to solve this problem exactly.

So now what we want to do is we want to use a numerical tool. And the gold standard method the numerical method for these kind of problems is so called RK 4 method.

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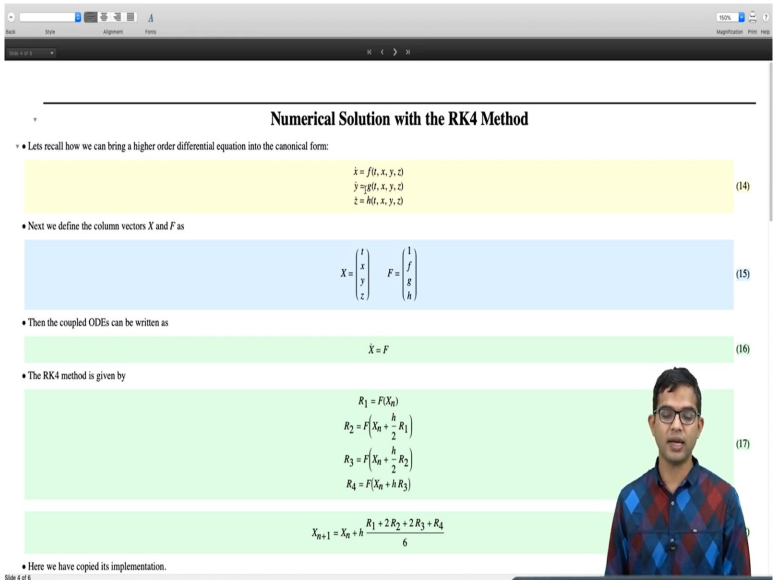

So this is a Runge-Kutta 4 method which is given by you know this set of equations we will just recall how this works out. So given a set of given a differential equation no matter what its order is there is way to bring it into this canonical multiple variable first order form. So you want to put it into this form $\dot{x} = f$, $\dot{y} = g$ and $\dot{z} = h$ and so on.

And it is always possible to do this and we have seen that it is convenient to define a vector which involves the time as well. So we can take the first variable to be time and then we have *x, y, z* and so on. And then you define a function a vector of function f where ofcourse the time derivative of time is just 1 so typically you will just have 1 as a first row of this column vector *f*.

Then you will have these function *f*, *g* and *h* and so on. And then there is compact way to write down this differential equation and no matter what the order of this differential equation is. So that is the duty of this abstract formulation so then you have $\dot{x} = f$ vector equation and then the RK 4 method is given by you know it is a combination of these ideas of Euler's method and improved Euler's method.

So we saw that in Euler's method you have this very crude approximation of what the value of derivative is. In the improved Euler method you use the crude estimate to go to the next step and then find the derivative of this sort of false next step. And then take the average of the derivative information at these 2 points.

And when you use this average information suddenly the method actually becomes much more powerful. And so, the people have played around with these kinds of you know taking averages so the even RK 4 involves something like this.

You take you see you have all these numbers $f(x_n)$ is like the derivative of the first point. Then $f(x_n) + h/2 * R1$ so you use *R*1 to go to a certain step ahead and then get derivative information from there. Then you get derivative information at *R*3 you know which is $f(x_n) + h/2$ using *R*2 and then you go to *R*4 using information of *R*3.

So there is a lot of theory on why you know these special numbers of *h*/2 and *h* * *R*3 and all these things have chosen. But that is not really the goal of this score is not really to understand all these why these special numbers are chosen but it turns out that it works and it works very well, right. So that I told you that this is a gold standard and there is a reason for this and it is possible to do thorough test.

And it is an art form for how you know perhaps originally work this out but there is also it is also a science because you can argue for y this method is superior to say the improved Euler method or you know other kinds of method.

You could also play with other kind factors here why *h*/2 instead of *h/2* you take some other α, $\boldsymbol{\beta}$, γ, $\boldsymbol{\Delta}$ and then these kinds of games have been played and RK 4 method is taken to be a, an excellent method for this and it is the gold standard. Okay so we will just borrow all these input.

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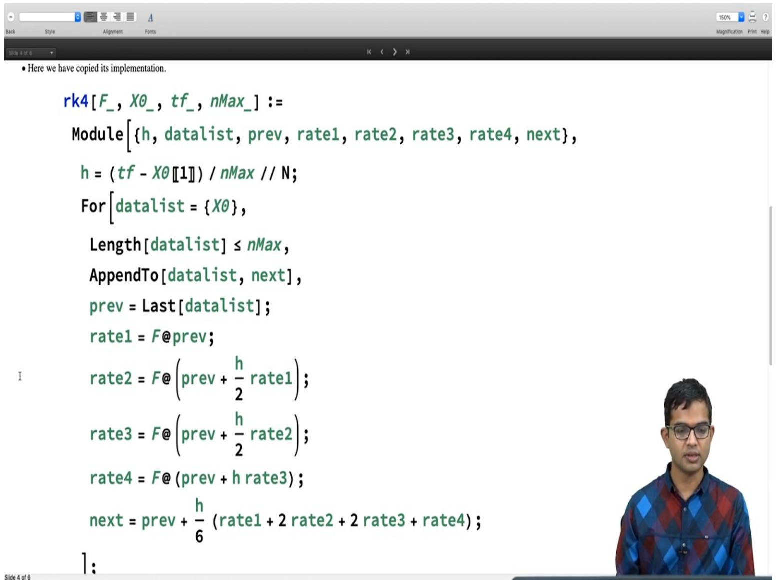

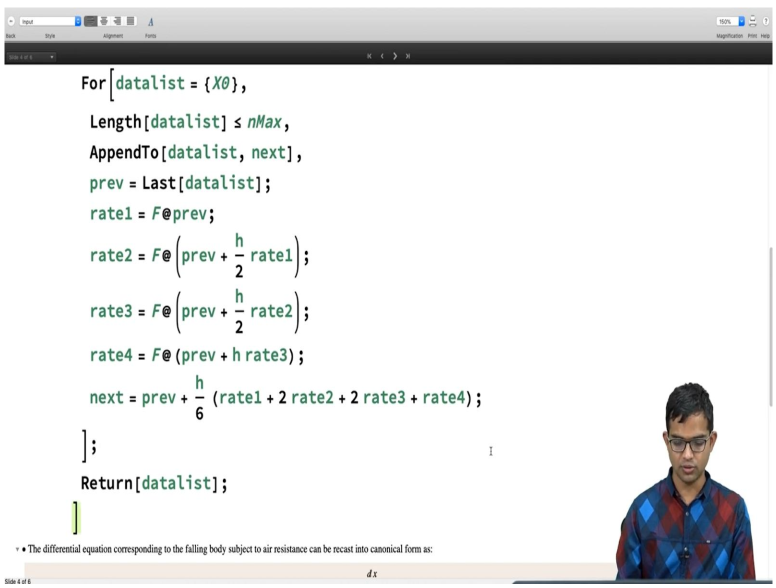

And then we have in fact the code also written that is a very nice compact code. And which has been discussed in a previous video, if you have not checked this you will go back and the details are there. So there is a nice modular way of written this is crude way to writing which you can start with and then you up and you make use of all these functions like AppendTo and you know where all these rate 1, rate 2, rate 3, rate 4 are all taken care of in a very nice efficient way.

So I will just borrow this code from there and then I have run this code just for Mathematica to know that this that I have loaded it in some sense, right its what I have done.

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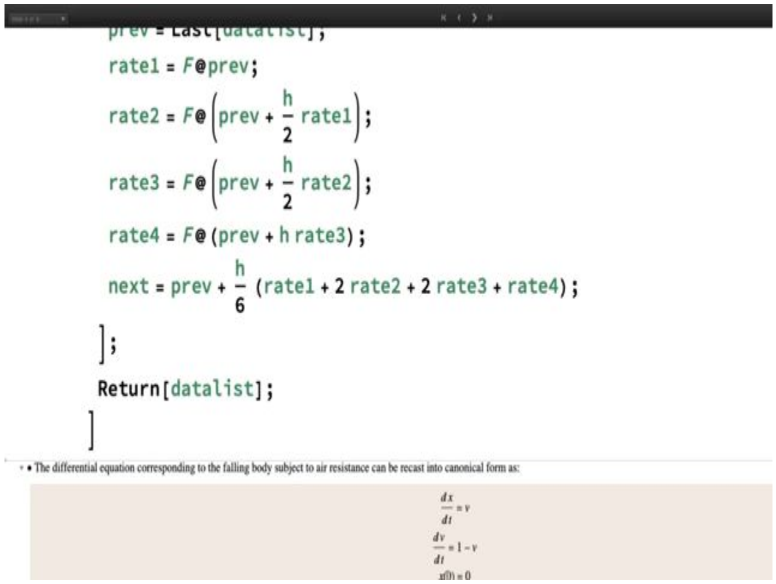

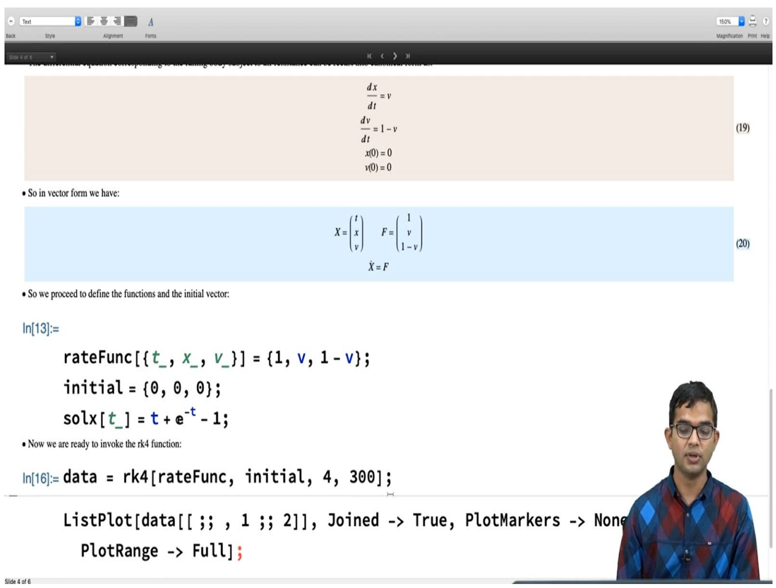

And then I have written my differential equation also in this canonical form so $dx/dt = v$ I have already used this then I have $dv/dt = 1 - v$. So that is all I have just 2 variables in this problem a very simple problem I have an analytical solution as well.

So then I have $x(0) = 0$ we have 0 to 0 so in vector form I have capital X is given by this t x and v and capital F is 1, v and 1 - v. These are the, it is the derivative vectors. So then I have $x \neq f$ so we proceed to define the function and initial vector.

So the rate function is going to be a function of t x and v and it is basically mimicking this vector f here that I have 1, v and 1 - v I have already put them in. And then the initial values are time $t = 0$, $x = 0$ and $v = 0$. And then I already know the solution to this problem so I have just called this as a function $t + e^{-t} - 1$ so that I can compare.

So now I will go ahead and invoke this RK 4 function our own RK 4 function and then specialized to this particular problem I go ahead and run this I am running this only up till 4. And you see that the number of steps involved is also very-very small much smaller than what I needed with Euler method.

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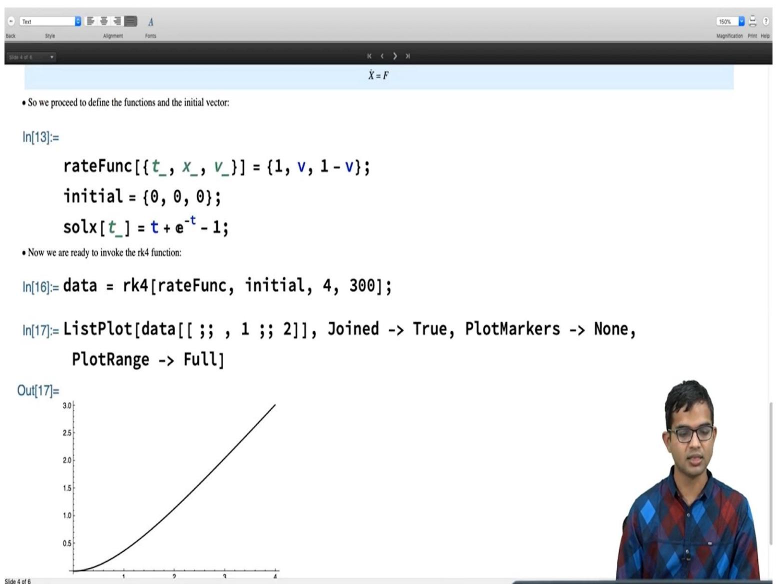

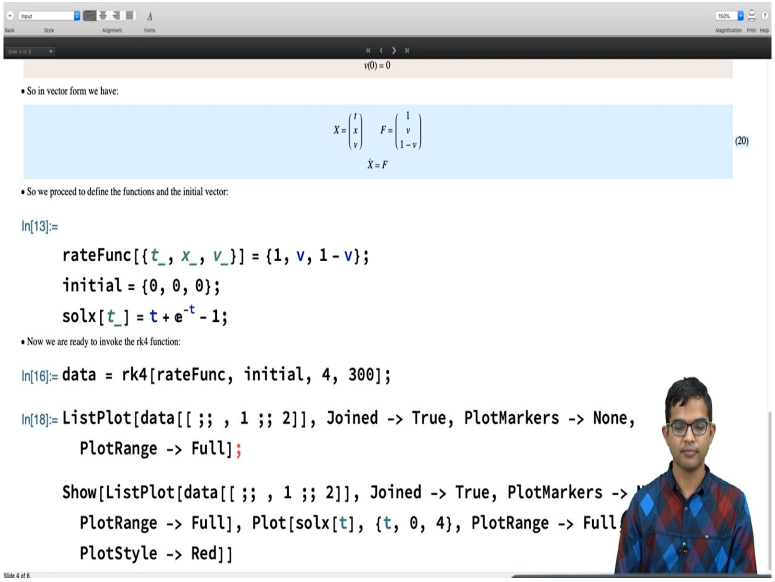

And then I go ahead list plot only the data which has come out of this and it looks promising so it is giving me information about *y*. I mean I called it *x* here but it is really the distance covered by the particle as a function of time. So of course I need to go ahead and compare against my old data and here I have *d1* that. So this is the code which compares my numerical data with the analytical solution.

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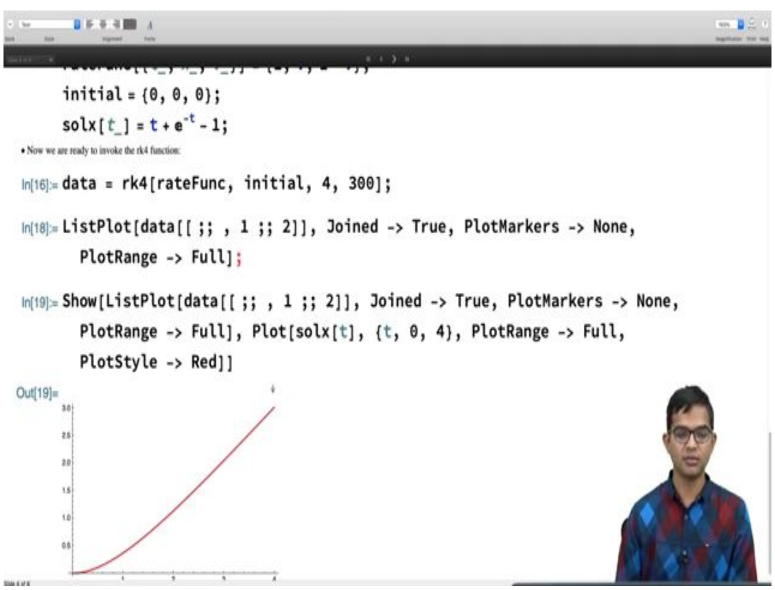

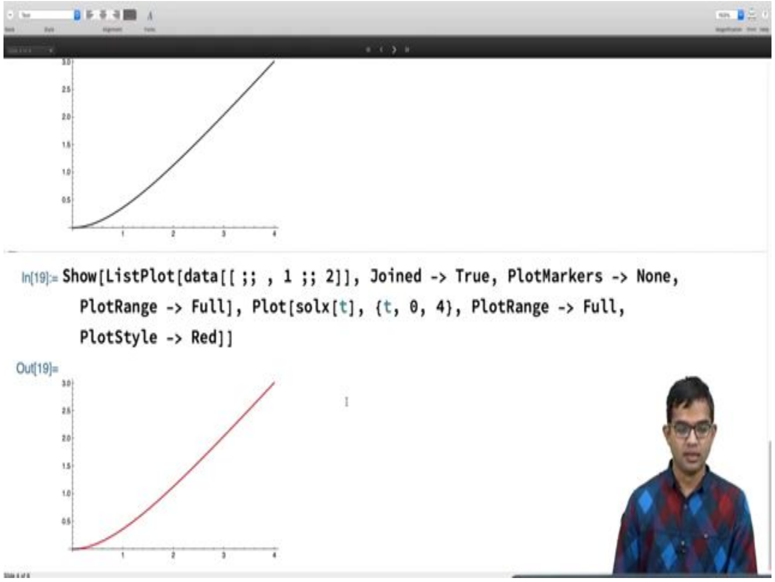

So you see that the 2 curves are basically completely indistinguishable I cannot even see that there is a black curve sitting behind that. So the black curve was this and then alongside I have also plotted this solve $x(t)$ which is the analytical solution. It is gratifying that my numerical method using RK 4 with just 300 steps which is like superfast it just finish its run in no time and when I am evolving up to time step 4. Suppose I go to even larger I can do that.

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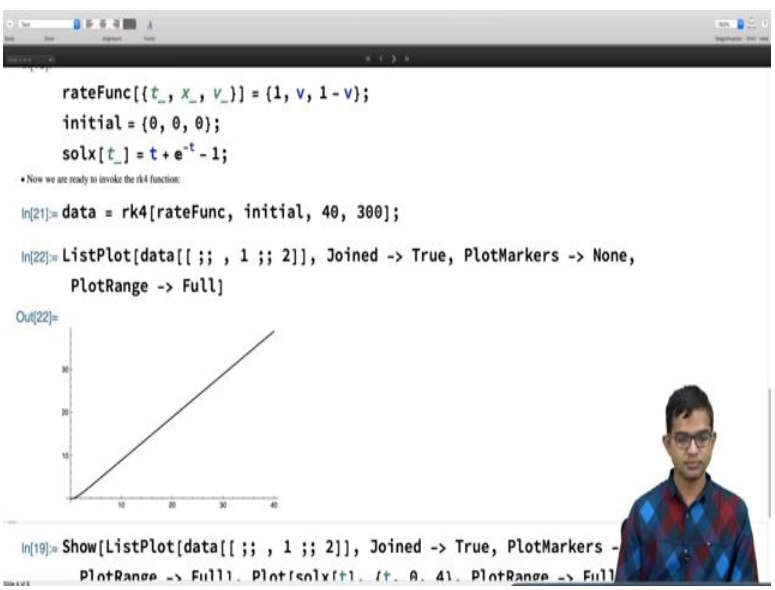

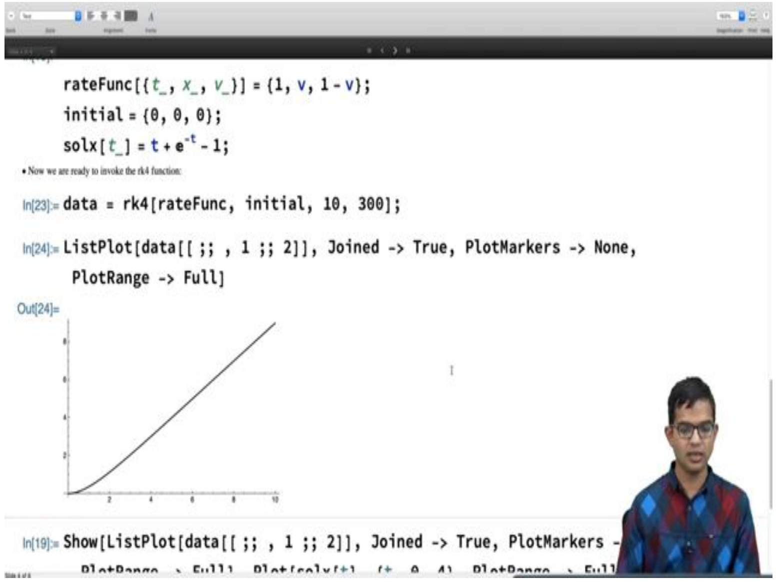

So if I make it 40 then if I run this then I have it go. So the reason I do not want to do this is because it is going to completely obscure of this initial part and it is going to make it look as if it is completely linear. So it is better to keep it only up to may be 10 let us keep it up to 10. I will plot this there you go this curved region at the start and then it becomes linear.

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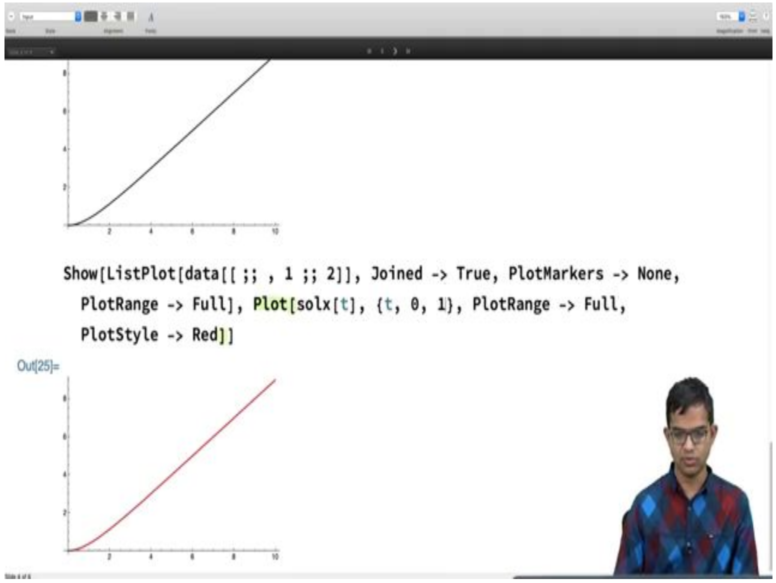

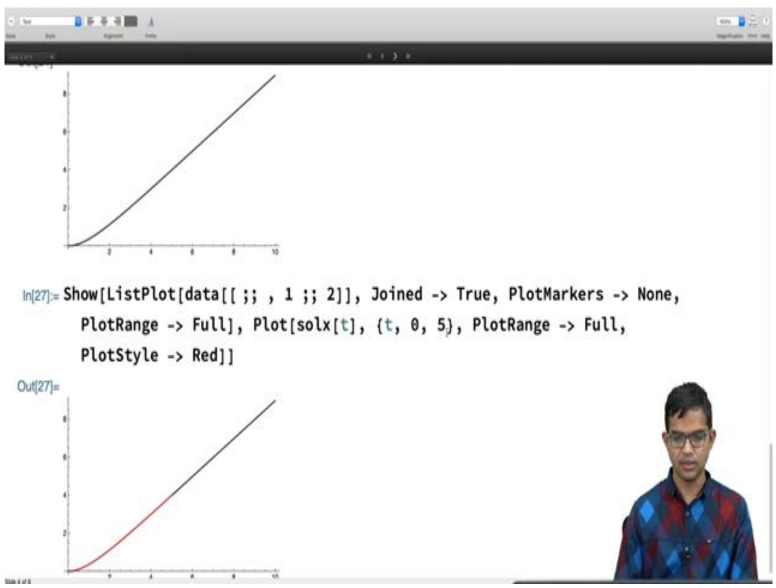

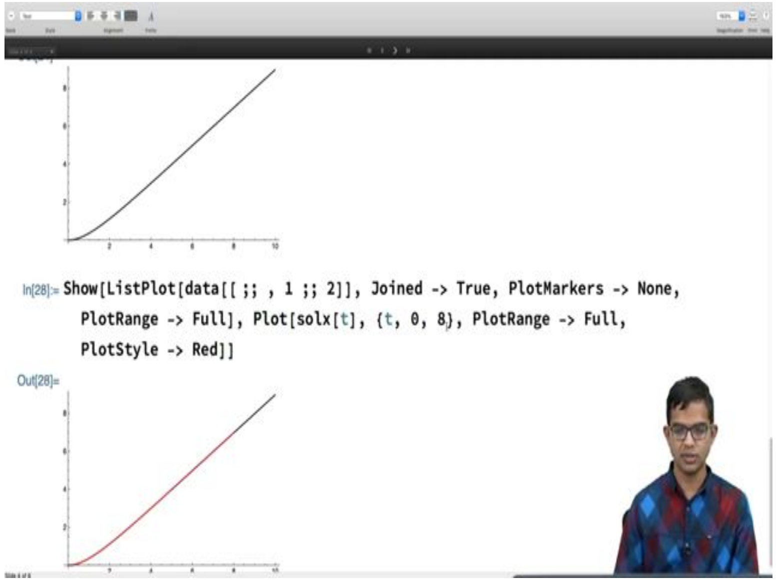

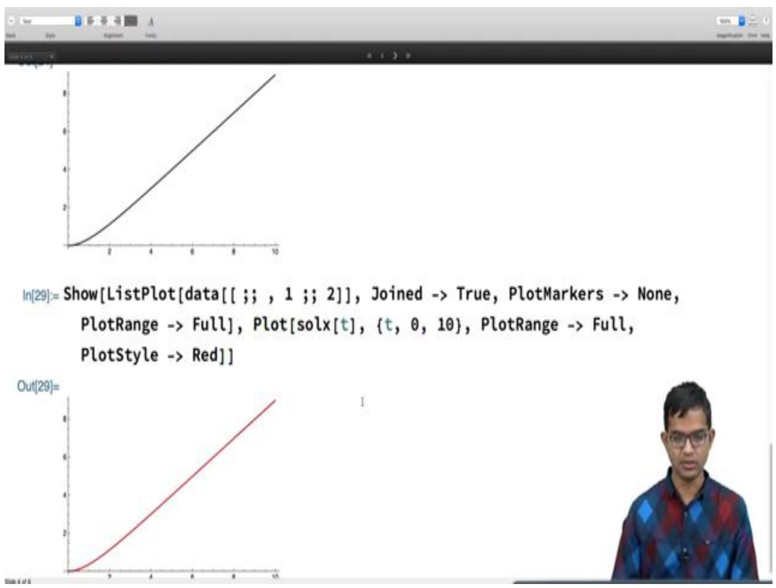

So here also I will run it up to 10 and there you go, so the two of them agree excellent. So if I make this run up to 4 then you see that the red curve is right on track. And in fact I can keep on increasing this 5, it is going up and then I can make it 8, it is going up and then indeed the 2 curves lie right on top of each other very gratifying.

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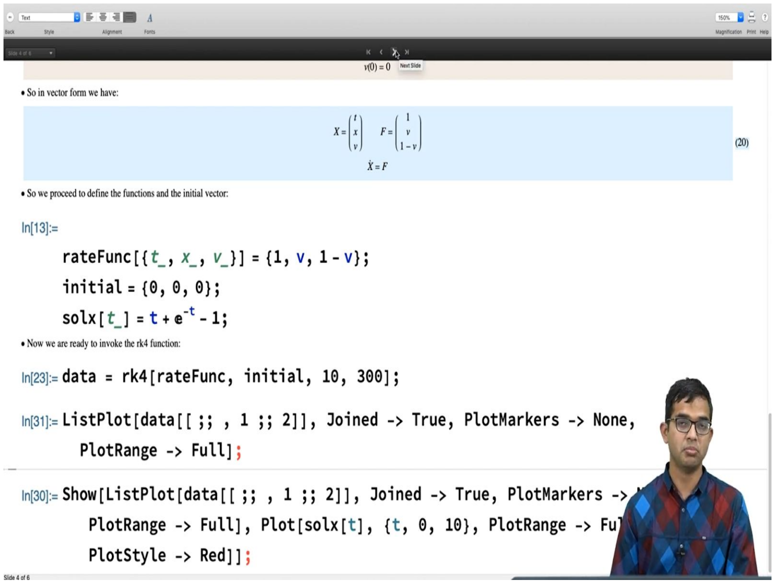

Okay so now what I want to do is I want to consider a variant of this problem I have considered a relatively simple problem although we have frictional forces involved it is simple because it is analytically solvable.

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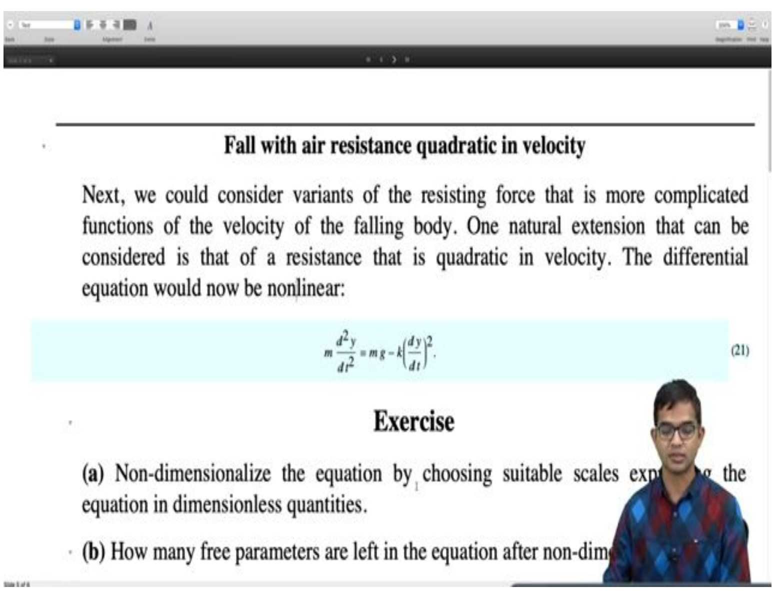

Okay so let us consider a generalization of the previous problem so here I am going to take the air resistance I still have air resistance but where the resistance is now quadratic in velocity. So last time we had linear in velocity. So this is not a not a completely different problem and yet already it is inherently a different beast altogether you are dealing.

Because it is a nonlinear differential equation now but it turns out that this problem too is analytically solvable. So we will go ahead and do this exercise, so it is non-linear. So now the k that I have here is a must be interpreted slighted differently from the k I had earlier.

So this k is, goes with $(dy/dt)^2$ so $k*(dy/dt)^2$, k times velocity square has the same units as m g. So the non-dimensionalization here is similar to the previous one but also quite different.

So I urge you to pause the video at this point and go and carry out the same type of exercise we did earlier and extract the non-dimensionalized equation for yourself. And then find out how many free parameters are left in this equation that is the first exercise.

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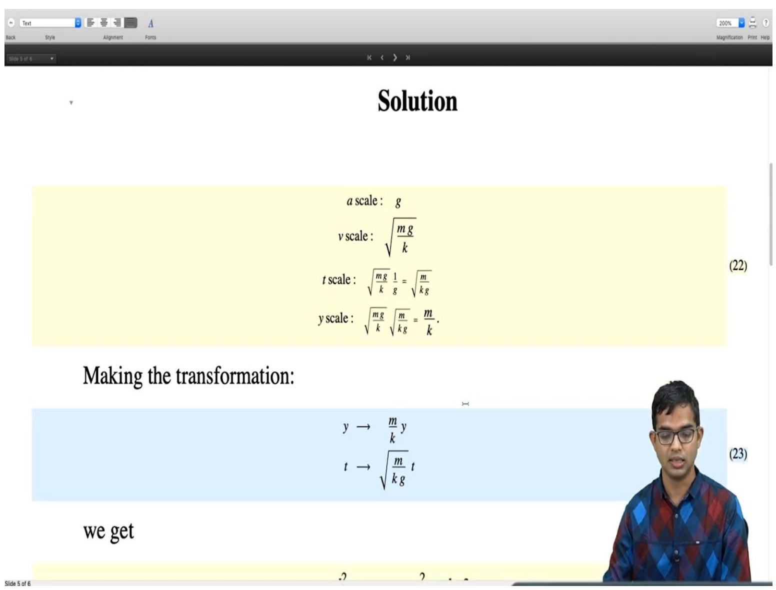

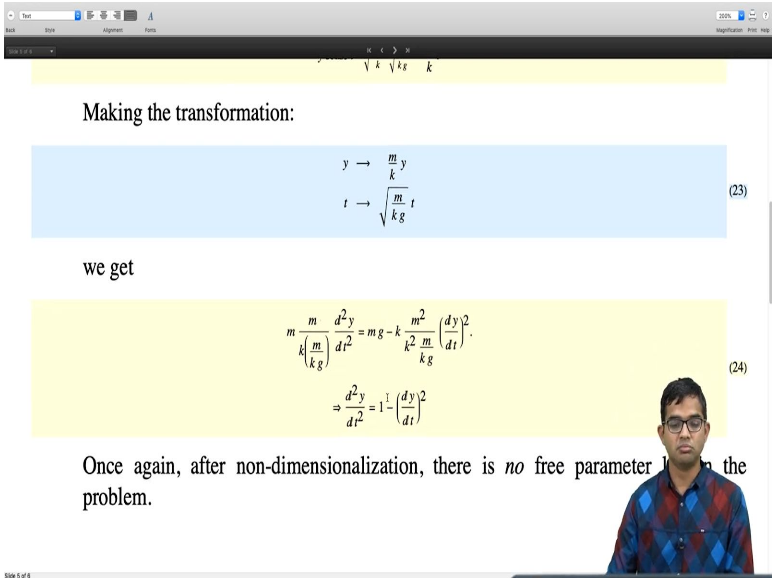

So I am going to show you my solution my solution is the following. Once again I start with the acceleration scale which is easy because gravity g is the acceleration scale. Clearly so $m\,g$ must have the same units as $k * v^2$. So $m\,g/k$ has units of velocity square so $\sqrt{m\,g/k}$ has units of speed.

So $\sqrt{m\,g/k}$ will be my speed scale, so then from which I can extract the time scale which is just given by $\sqrt{m\,g/k} * 1/g$ which gives me $\sqrt{m/k\,g}$.

So now therefore, a final step is to just multiply the v scale and the time scale and extract the distance scale which will give m/k. So once, I have this so the standard trick is to replace y by $m/k * y$ and t by times scales, times t which in this case is just $\sqrt{m/k\,g}$.

So then we have here all this simplification and so, I will get $m\,g * d^2/dt^2 = m\,g - m\,g * (dy/dt)^2$. And then in end I left with $d^2y/dt^2 = 1 - (dy/dt)^2$.

Once again there is no free parameter left in this problem. So the physics basically does not change the qualitative nature of the solutions just do not depend upon any of these parameters m g k are, just you can choose whatever values you want they will only change some numbers. The numbers will be different based on what m g and k are.

But the essential physics does not care about any of these numbers. So that is the advantage of doing this non-dimensionalization. It gives you the concentrated form in which you have removed all the unnecessary stuff and keep only the essentials and then of course you have to go back and interpret. So that comes in the later stage when we also compare it against numeric.

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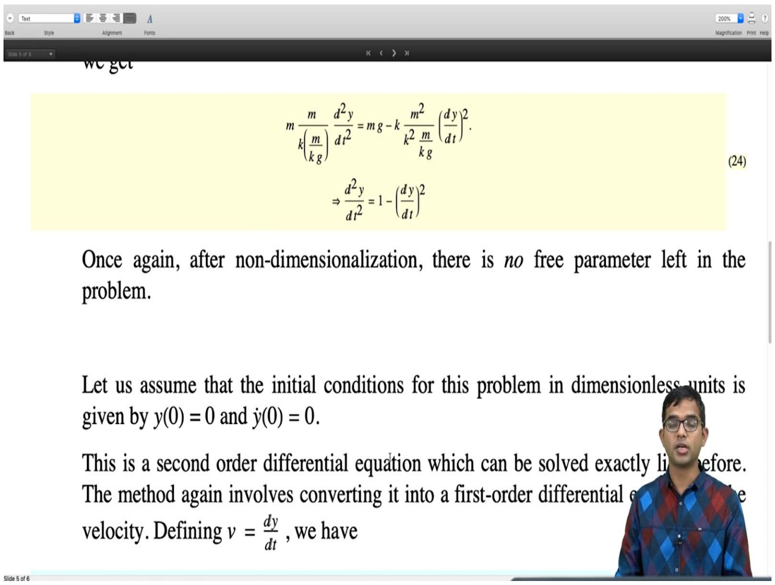

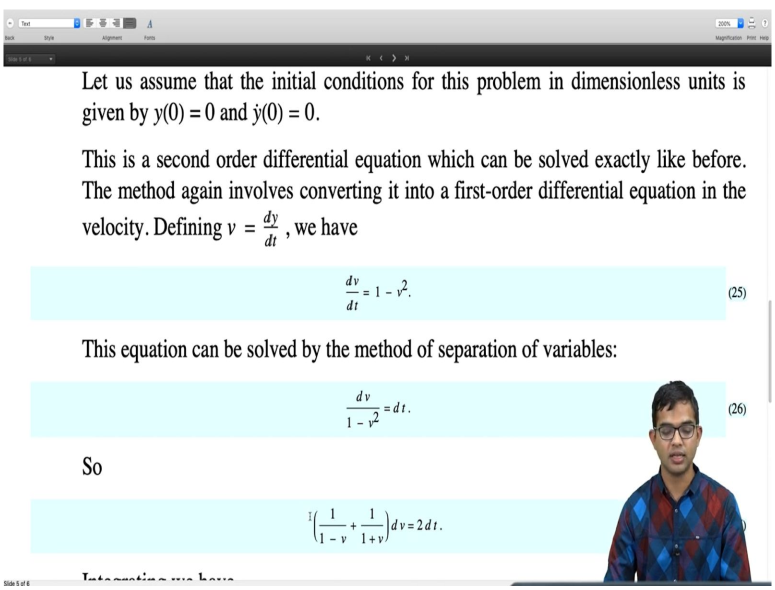

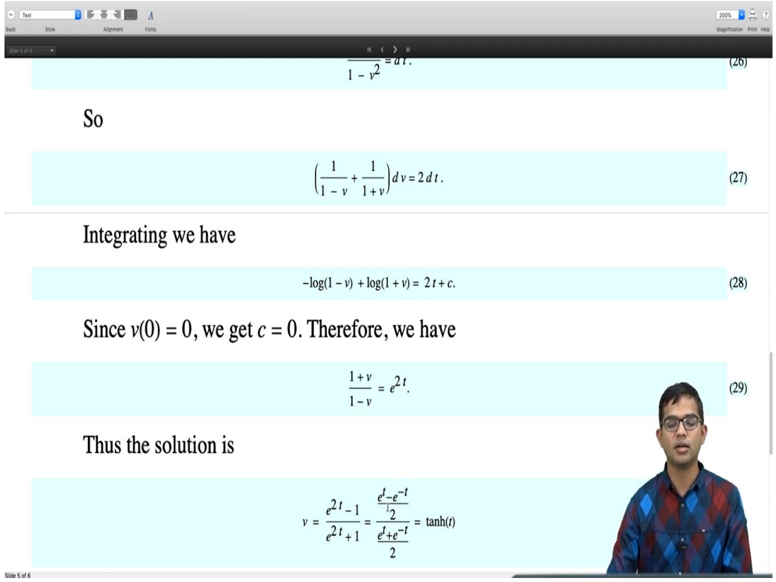

So once again, I am going to assume initial conditions to be $y(0) = 0$ and $\dot{y} = 0$. This is a second order differential equation like before but crucially it is non-linear second order differential equation. I have $(dy/dt)^2$ and still this problem is special enough that I can still solve it analytically.

So I have once again the trick involved is to introduce a new variable $v = dy/dt$ and then reduce it to first order differential equation. It is been reduced from a second order nonlinear differential equation to a first order nonlinear differential equation. So I have $dv/dt = 1 - v^2$ and this equation can also be solved by the method of separation of variables.

I have $dv/(1 - v^2) = dt$ and then I have to write as a sum of partial fractions. So it is easy to see that it is just $1/(1 - v) + 1/(1 + v)$ and then I have pulled out this factor which was lying in half and take it to the right side. And then I just simply integrate on both sides I am going to get $-log(1 - v) + log(1 + v) = 2t + c$.

So if put $t = 0$ I need v also to become 0 and this will happen if this constant c is 0 and then a little more algebra and I have $1 + v/(1 - v) = e^t$. Thus the solution is $e^{2t} - 1/e^{2t} + 1$. You should go back and check these calculations.

And then you can rewrite this you know this numerator by denominator by rearranging certain things multiple throughout by e^{-t}. And then multiply the numerator and the denominator both by factors of ½ . Then you see that this is like *sinh*(*t*)/*cosh*(*t*) which is just *tanh*(*t*), that is the solution.

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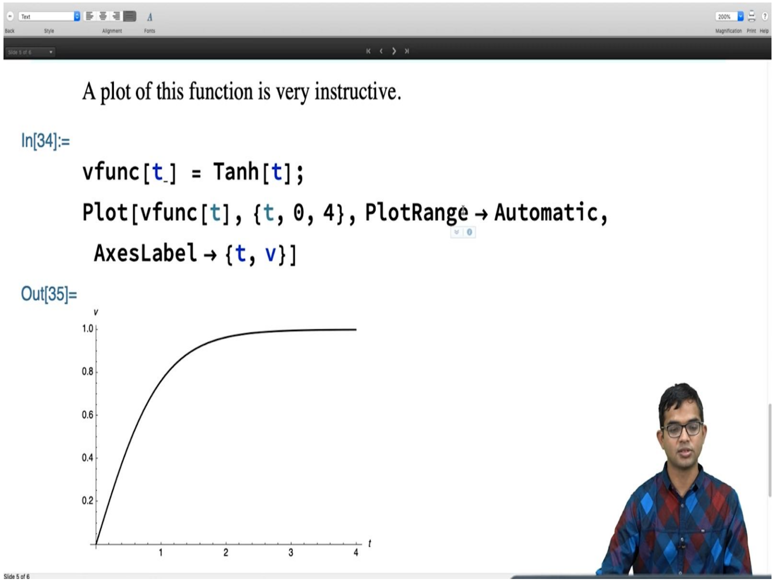

So the plot of this function will tell us that for large times once again you are going to get saturating behavior. So it does not matter whether your damping force, your frictional force is proportional to v or it is proportional to v^2. You are going to get you know v to saturate at 1. The only difference is of course what this 1 means, 1 is no longer 1 is now to be measured in units of $\sqrt{m\,g/k}$. So it is going to a value of $\sqrt{m\,g/k}$ that is all.

Otherwise and also that there are some details in which it differs it is the *tanh* function now earlier we had some e^{-t} and you know $1 - e^{-t}$ now we have a *tanh* function, okay.

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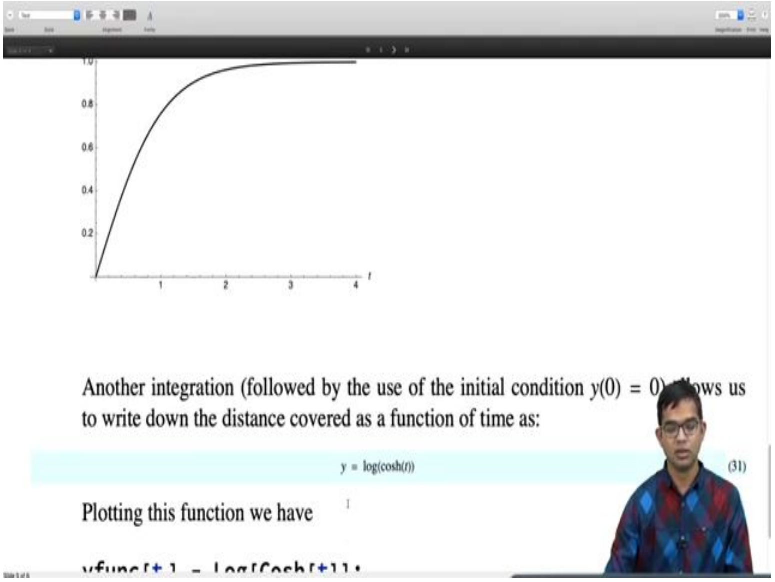

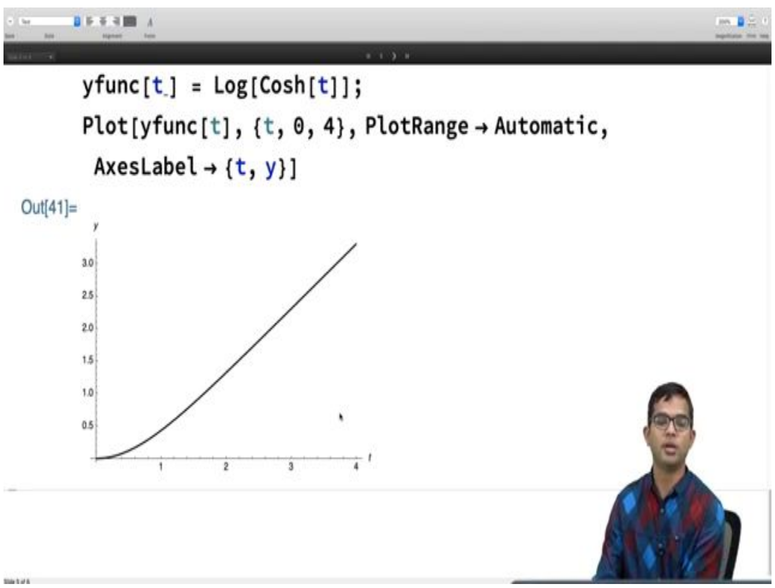

So it is also possible to integrate this out and write down the full solution for the distance otherwise it is going to be a constant and that constant is 0 in this case. So its $y = log(cosh(t))$ if I plot this, these 2 function also looks similar to my I do not need this if I do this and then plot this once again it looks like somewhat similar to my previous example.

So it starts from 0 and then keeps on increasing it bends initially and then after sometime it stops bending it becomes a straight line eventually. So $log(cosh(t))$ so it is kind of amusing that does not regardless of whether the speed is whether the frictional force is proportional to the speed or the speed square. You are getting solutions which are qualitatively quite similar.

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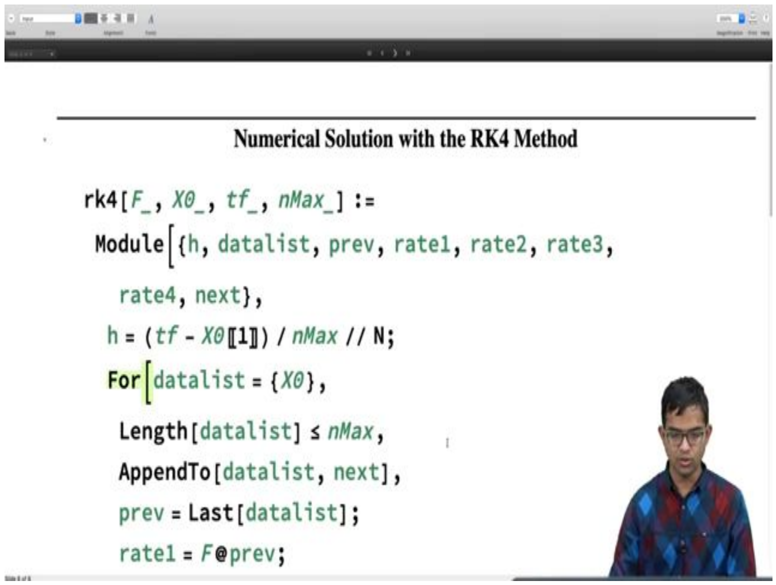

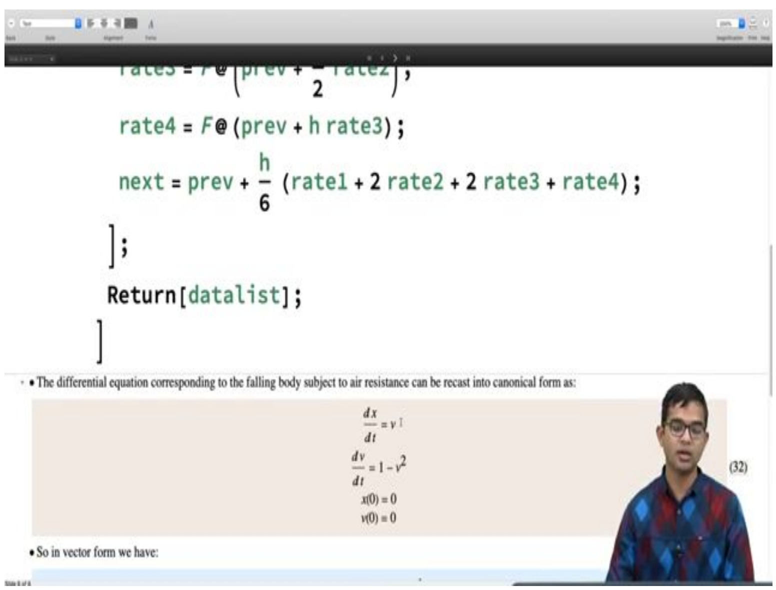

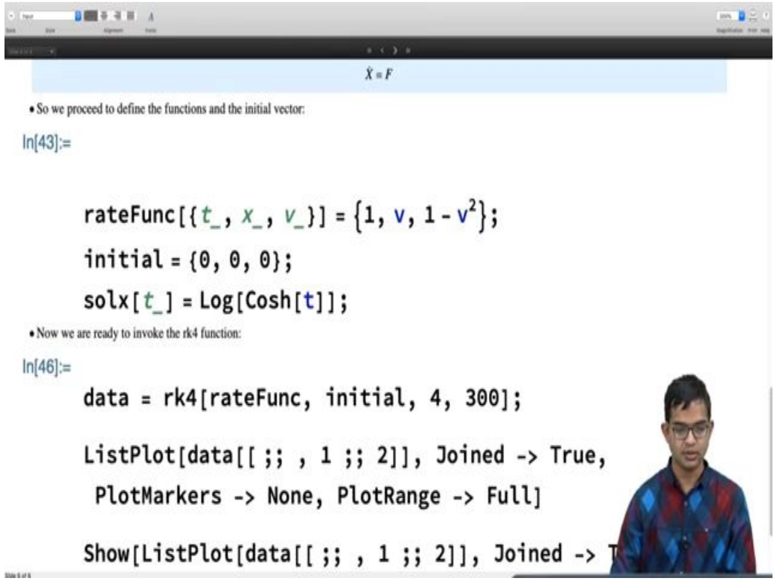

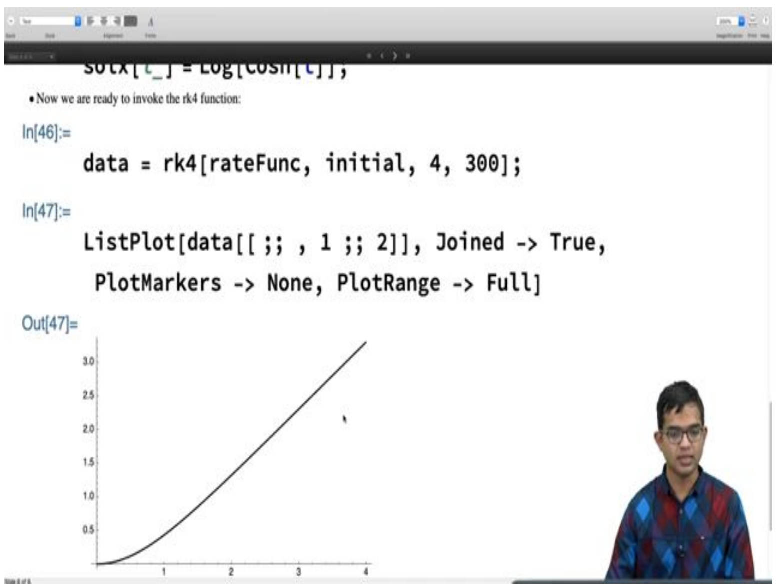

So let us check this with the RK 4 method. So we load the code and then it is already loaded but anyway I am doing it again just to be safe. So I have $dx/dt = v$ now and $dv/dt = 1 - v^2$, $x(0) = 0$, $v(0) = 0$. Now I have these vectors for x and f and I have $\dot{x} = f$ like before.

And now I have rate function is given by 1, v and 1 - v^2 , initial vector is the same solution is now $log(cosh(t))$. So I will hit shift enter here and now we are ready to run this RK 4 code on this problem which I do and then I will plot the output. So I have once again, the data looks very similar but we know that the details are different so this is the data for the position as a function of time, the position of the function of the time.

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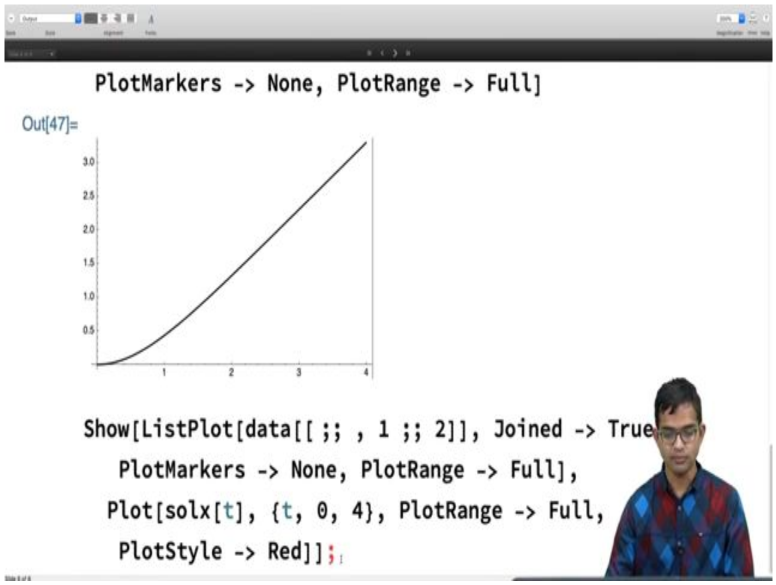

And if I plot, compare this against my analytical expression once again there is no way to distinguish between the 2 curves. And in order to distinguish I have to now plot one of them to a smaller value there you go.

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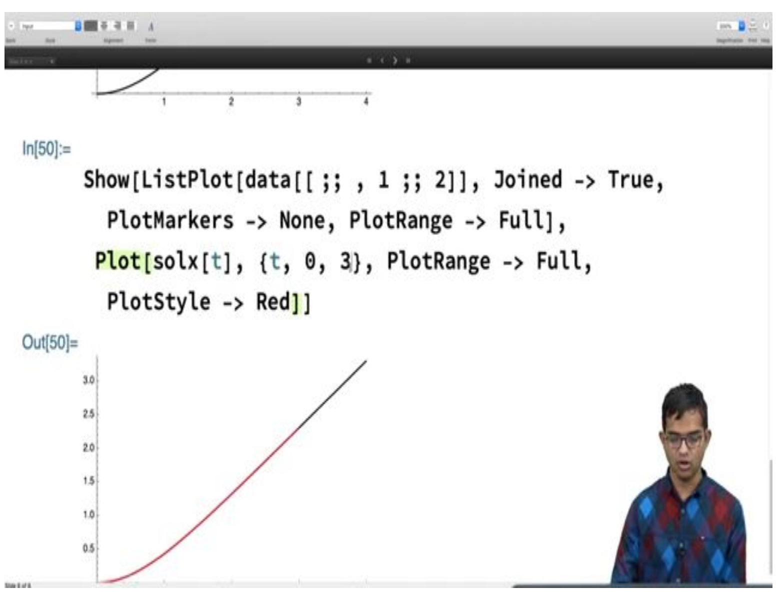

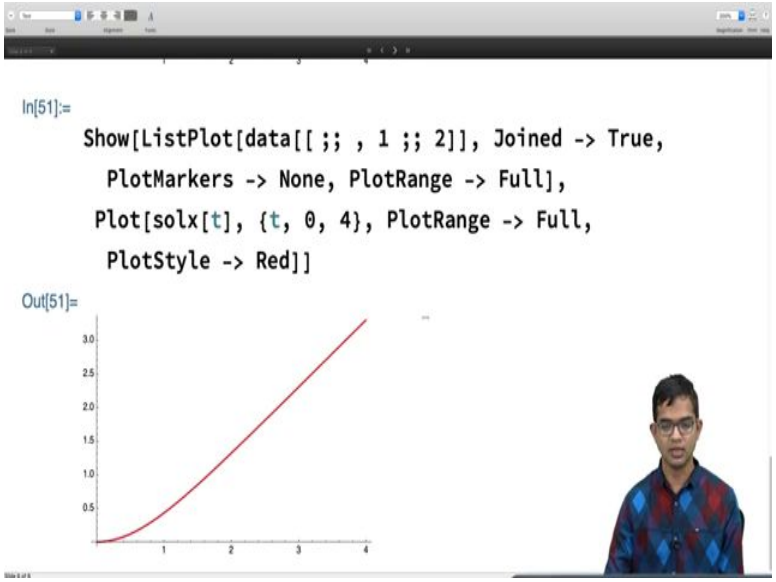

And you see that if I keep on increasing this it stays on codes and it is absolutely exact. So the numerical solution and the analytical solution are sitting right on top of each other. So this builds a lot of confidence.

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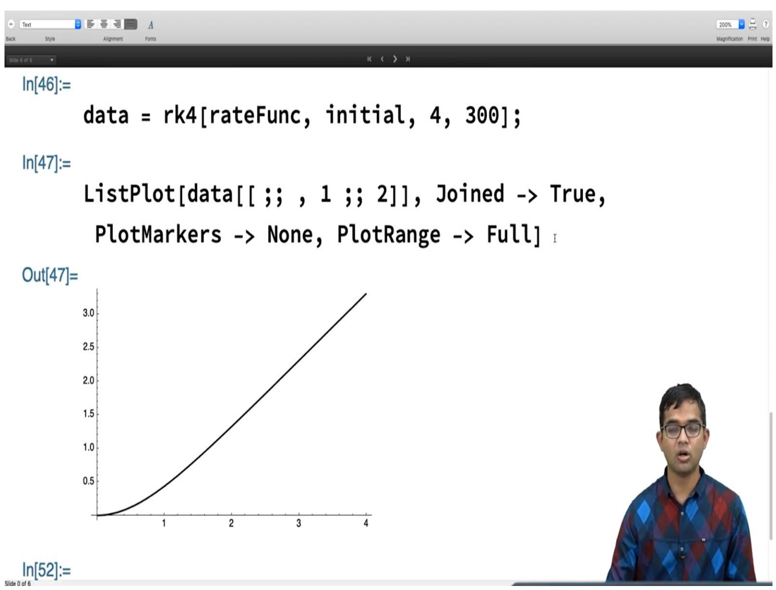

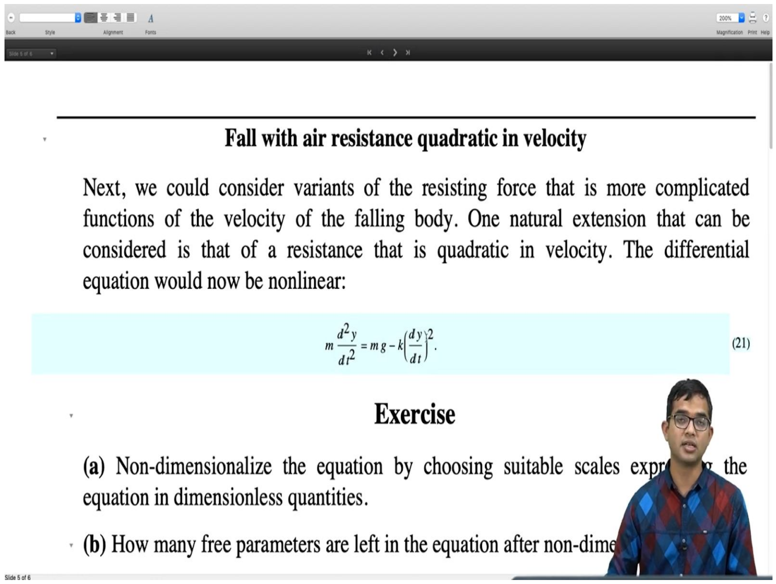

And so in fact what we can do is now we have built a very solid method by which we can play we can play more games we can look at more general situations where suppose you have drag forces which are some more complicated functions. What if you had sinusoidal function what you have, if you have an exponentially decaying function.

You know you have a region where if it suppose it operates only in a certain region let us say that you have a you drop a ball from the top of a tall tower. And there is you know from the 10^{th} floor to the 5^{th} floor the air is thinner so perhaps the drag forces are not so important. So you can come up with the function where the drag forces are tuned to different regions in space.

And then analytical solutions may not be possible and then you can just use this method and then check whether eventually you will get a terminal velocity or not so this is what is called a terminal velocity. So this is one thing you can do.

Another problem which is worth trying out is happens if your gravity is not a constant so we have assumed that we are in this problem we assumed that we are sufficiently close to the earth such that you can take the force acting upon the particle to be just $m\,g$.

But we know that actually Newton's law of gravitation tells us that the force is going to be $G\,M\,m/x^2$, where x is the distance between the particle and the center of the earth. Now if this particle is sufficiently closed to the surface then it does not matter whether it is x or you know slightly more it is always the radius of the earth plus some number. And so for all practical purpose you just take to be just the radius of the earth.

But if you are going to drop this from a really a large height where it is some significant percentage the height is significant percentage of your the radius. Then you actually model this differential equation completely differently. So this is also some kind of a game that we can play or if we have some type of astronomical type of phenomena involved.

Suppose we want to send out a satellite into space and then we definitely have to consider these kinds of equations. And then there would also be drag forces involved in more complicated functions of v they can all be games that you play out. So for simplicity you can start with you know what if you had a fourth power instead of second power.

If you had third power what would happen and so on. This is a very easy modifications you can make to this differential equation and then you can come up with more complicated versions. Okay, hopefully you got something out of this module and it will inspire you to play more games. Thank you.