Physics through Computational Thinking
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Lecture 30
Driven Oscillations

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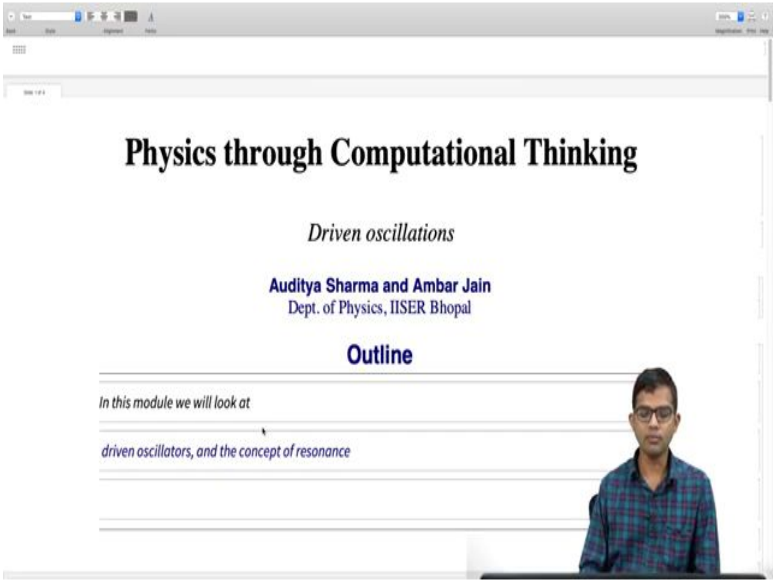

So, in this module, we are going to look at Driven Oscillations and understand it also with the help of a numerical method, which is known as the Euler method. So, I will assume that you have already seen the Euler method. And so, and yet, I will try to make this discussion as self-contained as possible, right.

So, there must be a way for you to go back and check out this video, where we discuss the Euler method and where we work out the numerical tool, which I am going to just use this tool, as if we have already done it. And yet, I will try to make this module also as self-contained as possible.

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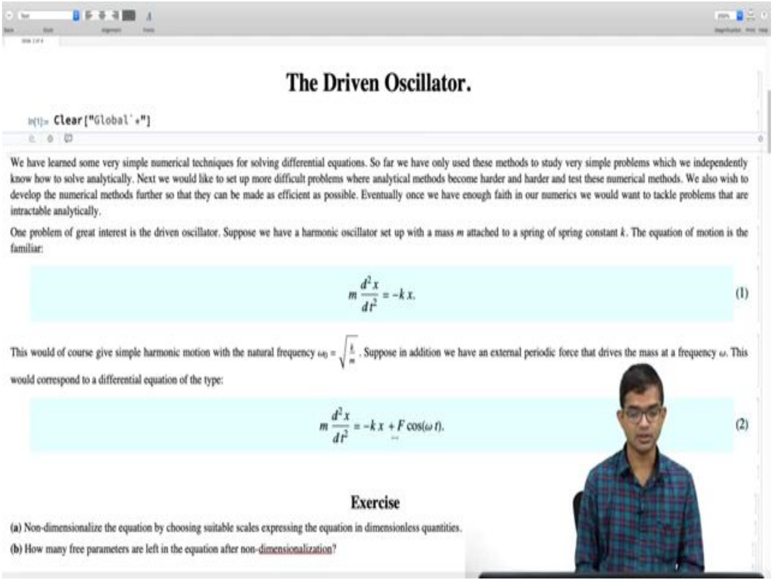

Right, so at this point we have already seen some numerical methods for solving differential equations, right. So, if you have skipped ahead, then perhaps now is a good time to go back and check out these videos. But even, otherwise we will quickly briefly mention, what is what is happening here. So, what is the Driven Oscillator?

So, the Driven Oscillator is a simple harmonic oscillator, in which there is a forcing term. So, we imagine this is the simplest, a simple harmonic oscillator. The simple harmonic oscillator is given by just this differential equation, $m\, d^2x/dt^2 = -k\,x$, right.

So, you imagine that there is a spring connected to a mass m and the spring constant is *k*. And so, we know that it is this mass time acceleration is the force, given by $-k\,x$. And the differential equation corresponding to this is $m\, d^2x/dt^2 = -k\,x$.

And from high school, we already know the answer to this type of differential equation is just given by $x = a\,cos(\omega\, t)$, ω_0 I have defined $\omega_0 = \sqrt{k/m}$ here. Omega naught t plus, there would be both sine and cosine, right.

So, this is something that we were told is a solution. I do not think, unless you have already taken somewhat more sophisticated course, so, you perhaps do not understand how to work out the solution for a general differential equation of this kind. So, the theory of differential equations, particularly of this kind is quite well-developed.

So, there are systematic methods, by which one can actually arrive at the answer to this kind of a problem. And particularly, differential equations, which are linear in nature like the one we considered here, it is linear, because there is no square in x, right or this d^2/dt^2 only appears as a linear object, it is not $(d^2/dt^2)^2$ or there is no nothing like $sin(x)$ or no complicated things involving x.

But you can have, you know, as complicated function in t, as you want. And still it can remain linear. And one very important case of this kind is a driven oscillator. So, you imagine that you have your, you know the mass m, which is attached to your spring, which is being kicked after every, by some external force where you can think of a $cos(\omega t)$, but you can also imagine; you know, giving it some impulse every, you know, so many seconds.

So, it can have a time period t associated corresponding to which there would be a frequency. So but, suppose you imagine a cosine forcing function. You can have a square wave, you can have a cosine wave, you can also have these delta pulses, you know all these kinds of problems, one is interested in, one is you know it is easy to motivate directly from a physical perspective.

And these kinds of problems are also very closely related to circuits. You know circuit phenomenon, where you have an voltage and some resistive element, capacitive element. So, you can have LCR circuit, right and which, each of these components of your circuit would perform roles, which will eventually come down to a differential equation of this type. And you have an external voltage.

And that voltage could be quite easily designed to be an AC voltage. And then you would get something like $F\ cos(\omega t)$, right so, once you have solved this problem from a mechanical point-of-view, it would be very straightforward to work this out on a circuit as well, right, which is something, I will urge you to do and perhaps there would be a homework of this kind.

And so, it could also well be that, my colleague Ambar would describe some of these problems in the circuit language. And I might give you the mechanical version of it just so that, you see different flavors, but basically it is really the same differential equation and you should be able to spot either of these and realize that really they are manifestations of the same.

Okay, so, the first step in dealing with differential equations of this kind is to do a non-dimensionalization, right. We try to emphasize this repeatedly, that it is useful to take a differential equation and tear apart; you know all the stuffs which is not essential, when you are putting it on to a computer.

Eventually, we want to put this kind of a differential equation on a computer and solve it numerically. That is one of the tasks, right. So, ofcourse, in this case, we will show you how in fact there is an analytical solution as well possible. So, but the first step is to non-dimensionalize the equation, by choosing suitable scales; expressing the equation in dimensionless quantities.

So, if you want, you can pause the video now here and try to work this out. I actually urge you to do this. And then, after you have done your own version of non-dimensionalization, you can cross check against the version that I have here, right, so then once you have non-dimensionalize.

So, right now you see that this equation has a mass m, there is another parameter k then there is another parameter ω. So, there seems to be lot of parameters, ω_0, I said was a useful quantity to define this equal to $\sqrt{k/m}$.

So, how many parameters would remain, after you have carried out this non-dimensionalization exercise, right? So this is an exercise, worth doing and I urge you to pause the video and carry it out yourself. So, my own solution is the following.

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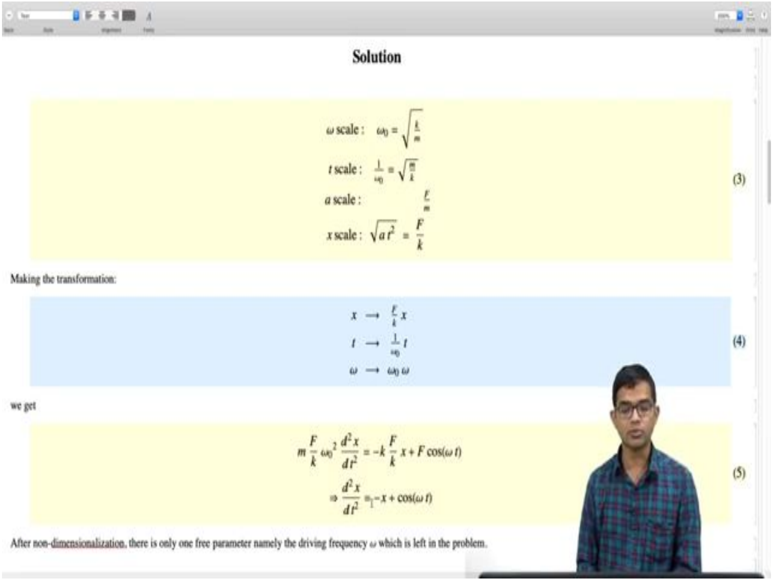

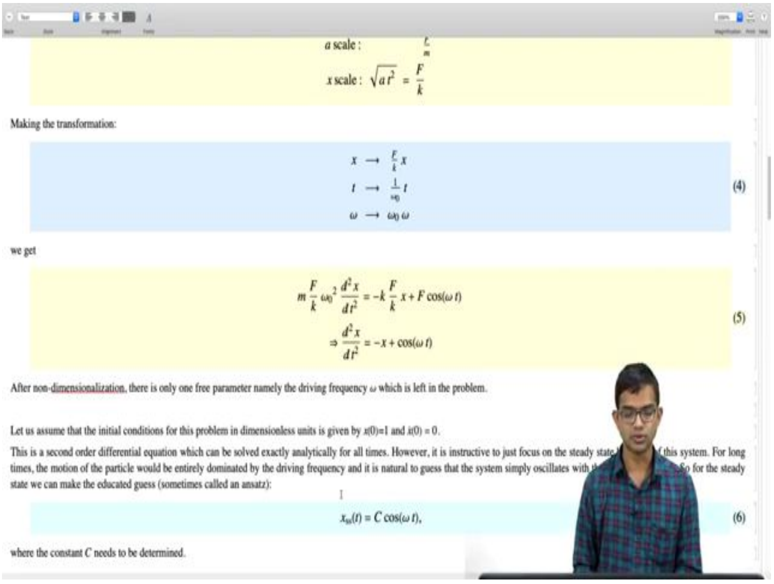

So, I spot that there is this ω scale. Right; $\omega_0 = \sqrt{k/m}$, it is the natural frequency of your oscillations, even when there is no external, external force being applied to the system. From which, we can derive a time scale. Time scale is just $1/\omega_0$. So, it is $\sqrt{m/k}$.

And then acceleration, right. Acceleration is something that comes from my force, the external force. The amplitude of the externally applied force is F. And I have this mass m. So, I have a natural acceleration scale, which is F/m. And therefore, I can actually get a distance scale.

So, distance scale is $\sqrt{a\,t^2}$, right, which I could have equally got by directly doing F/k, right. So, F is equal to, force is $k\,x$. So, I have these, all these scales. Some of these are actually derived scales. Some of these, they all come down from ω_0 and from this F.

And what I am going to do now is, go back to my original differential equation, equation number 2 and then non-dimensionalize it. So, the technique to non-dimensionalize is simply wherever you have a dimensionful quantity, you just replace that quantity by the scale times that quantity.

And then, lot of cancellations will happen and then you will end up with an equation, which has no, which has been non-dimensionalized. So, in place of x, I will put $F/(k\,x)$ in place, wherever I see t, I will put 1 by $\omega_0 t$ and wherever I have ω, I will put $\omega_0\,\omega$.

So, when I do this, I get my left-hand side is $m\,d^2x/dt^2$ will become $m\,F/k\,\omega_0^2$, d^2x/dt^2, right, so I have, so in place of x I have $(F/k)\,x$ and in place of t, in the denominator, t^2, right, in the denominator I have $1/(1-\omega_0^2)\,t$. So, it becomes on the left side $F/k\,\omega_0^2$, I have.

And likewise, on the right-hand side, I have $-k\,F/k\,x\ +\ F\,cos(\omega\,t)$. $\omega\,t$ will just remain $\omega\,t$, because you are going to replace ω by $\omega_0\,\omega$ and t by $1/\omega_0$ *t. So, they get cancelled. And now you have the final equation, which is just simply $d^2x/dt^2\ =\ -x\ +\ cos(\omega\,t)$, right.

And here, we have a non-dimensionalized equation right. So, these quan-, here you have a driving frequency omega, which is actually it does not have any dimensions. It is a dimensionless quantity. Now, this is a pure differential equation, which is a mathematical equation. And then you will have to; at later stage, once you have solved this, will have to go back and interpret what x means or in this non dimensionalized units, okay.

So, what is the problem we have? So, let us assume that we have to give also some initial conditions. Suppose we give the initial conditions for this problem, wherein (non-dimen) in dimensionless units, if you have $x(t)\ =\ 0$, is equal to 1 and the speed of your particle is at time $t\ =\ 0$, is also taken to be 0.

So, this is the second order differential equation, which can be solved exactly, analytically for all times. And we will talk about how to do this. But however, it is actually, it is useful to consider what happens to this, in the limit of very large times. So, this is what is called a steady state solution of this problem.

And to get to the steady state solution, we can make an educated guess, also known as ansatz, sometimes. And that ansatz is simply $C\,cos(\omega t)$. You try to mimic just the forcing term, so it is as if for a very large times, the system has forgotten its initial conditions in some sense. And it is only, it is the drive, which is causing it to closely mimic the drive itself.

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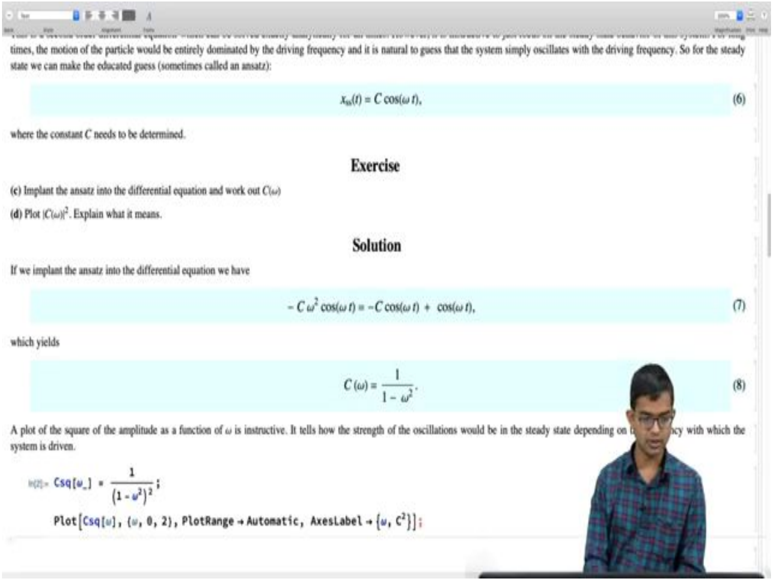

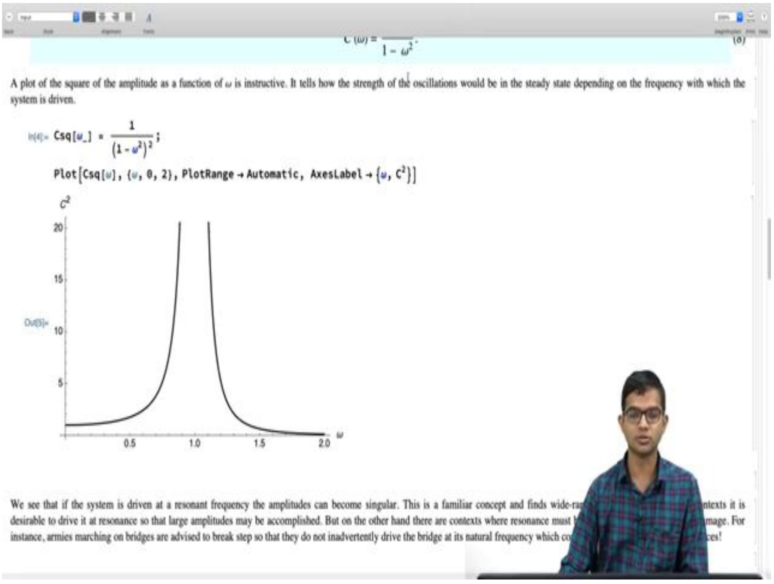

So, now if you directly plug in this ansatz into the differential equation, into equation number 5, can you extract a constant c and find out how it varies as a function or omega and make a plot of this using Mathematica and see what it means, right, so this is an exercise, which you should do before I show you the solution.

So, I urge you to pause the video at this point, work this out for yourself and then continue the video, right. So, here is my solution. If I implant this guess $C\,cos(\omega\, t)$, into the differential equation, then I have the left-hand side is simply, $-C\ \omega^2\ \ cos(\omega\, t)$. And then the right-hand side is $C\,cos(\omega\, t)\ +\ \ cos(\omega\, t)$.

Therefore, I can go ahead and solve for $C(\omega)$ and I get $1/(1-\omega^2)$. So, what does this tell me? This tells me that it is possible to find a solution of this kind and this will in fact turn out to be the steady state solution. And in fact, you see that if I were to buy this solution $1/(1-\omega^2)$ $cos(\omega t)$, you see that there is no connection of this solution to the initial condition.

It does not matter what my, where I was initially and what my speed initially was, the steady state solution will always be reached, because it depends purely on the external drive. So, we will discuss this in a moment. But let us understand what is going on with this coefficient $C(\omega)$.

So, the amplitude squared is of interest, it tells you of the strength of these oscillations. So, if I were to make a plot of this, this is what it looks like. I have written it down, okay let me do this. And then I plot it. If I plot it, so, I notice that there is something weird going on at $\omega = 1$.

In fact, the system totally, the amplitude totally blows up at $\omega = 1$. And if I pause to think for a moment, I realize that this is not such a surprise. So, what is going on here is, the concept is known as resonance, right, so this is a familiar concept maybe you have seen it in the course on waves or you know some oscillations type of course.

So, it is sometimes a desirable thing, sometimes something that you want to avoid. Okay, so here I have a plot of this amplitude square, so we see that it blows up at $\omega = 1$. And this is a signature of resonance. Resonance is the condition, where you know, if you drive your system at a very special frequency, which in this case is its natural frequency.

So then, somehow this is to the external force is conspiring, with the internal mechanism of your particle, you know at your mass m to make the amplitude you know, very-very large, right. So, resonance is sometimes a desirable point to drive our system at. At other times, you might want to stay away from resonance.

So, one example is you know that of armies which are marching across bridges they are often advised to not march in step, when they are walking along on a, on a bridge so as not to inadvertently become a driving force, which is by some chance, if the natural frequency of the bridge matches exactly with the frequency with which the army is marching on it, it may result in, you know a catastrophe, where the bridge may even collapse.

So, that is a context, where resonance is not desirable. It must be prevented. But there are other situations, where you want to drive your system at resonance, so that you can generate these high amplitudes, right, so let us move on.

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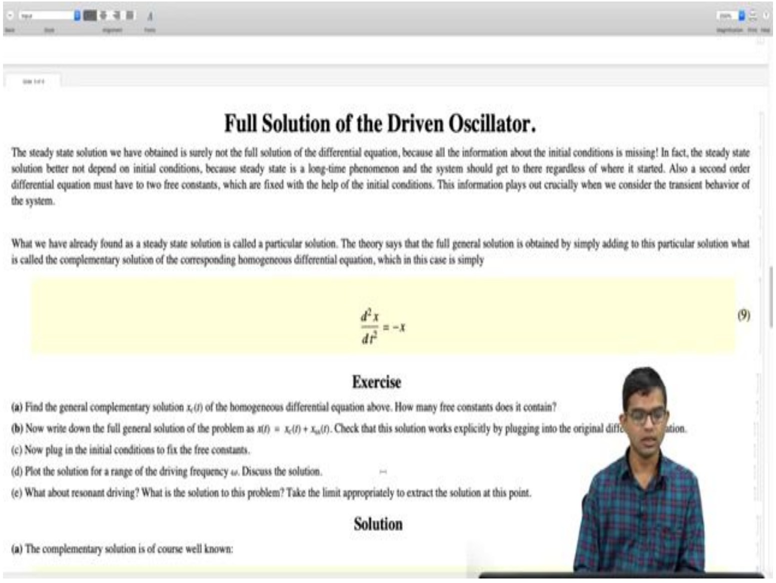

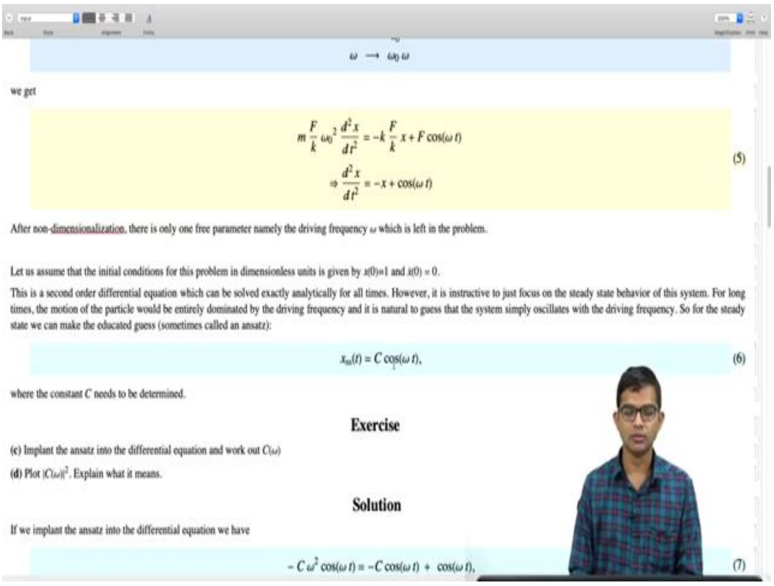

So, thinking about this whole thing as just a pure differential equation. So, the theory of differential equation, in fact tells us that there is something called a particular solution and a complementary solution. So, in this case, the steady state solution turns out to be a particular solution.

So, as you can see, it explicitly holds out right. If you choose your C to be this particular value, you can go ahead and plug back it into this equation. And then it is going to work out; no matter what time it is, right. So, this is an exact equality.

But often times, if you want, you want to get the full general solution, so this does not give you the full general solution, it is just a particular solution. If you want to get a full general solution, you must one way to do this and a very clever way to do this, is in fact to solve the corresponding homogeneous differential equation.

Homogeneous differential equation simply means there is no driving term. So, you take this original equation you had $d^2x/dt^2 = -x + cos(\omega\ t)$. And then you remove this driving part. So, then you are left with a homogeneous differential equation, which is relatively easier to solve.

In fact, in our case we know, we already know the solution. So, if you pull out the general solution of the homogeneous differential equation and you just simply add it to your one particular solution, then that gives you your, gives you the full general solution of the full problem, this is the theory, right and it is also intuitive why this should work out, right?

If you take any solution to this differential equation, $d^2x/dt^2 = -x$, then it is not a surprise, that if you can, you can just take a solution of this differential equation and add it to a particular solution. And it should still work out in other case, because you have this extra added term.

You know this part is going to, is only going to give you 0, right it is going to cancel out and the particular solution in anyway going to respect the full differential equation. So, if you are a little bit unconvinced about this, I urge you to explicitly take xp and xc which I will tell you what it is in a moment.

And then plug this back into your original differential equation and check for yourself, that indeed it is going to be a solution of the full driven oscillator problem, right. So, this is a standard and quite a beautiful method of solving differential equations of this kind.

So, what is a general solution of $d^2x/dt^2 = -x$? With that, we already know. And so, that is the complementary solution. So, once again I urge you to look at exercises a, b and c. And pause the video, solve for these before you proceed. So, if you make this into a habit, then your learning is is enhanced.

You actively try out something and then cross check against my solution. Perhaps you have an alternate way of doing things and maybe there is more learning, which comes out when you have multiple approaches. Okay so in fact you have, there is a, b, c, d, e all of which you should try out. Pause the video now try out a, b, c, d, e and then look at the solution.

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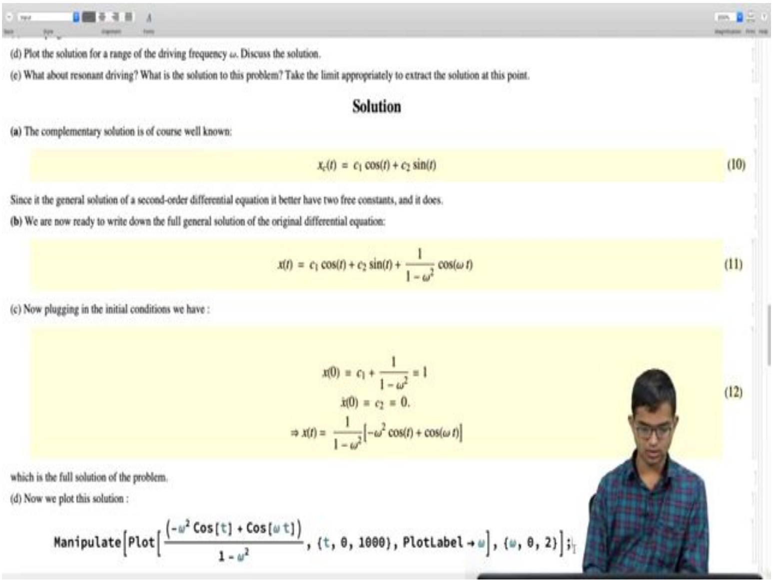

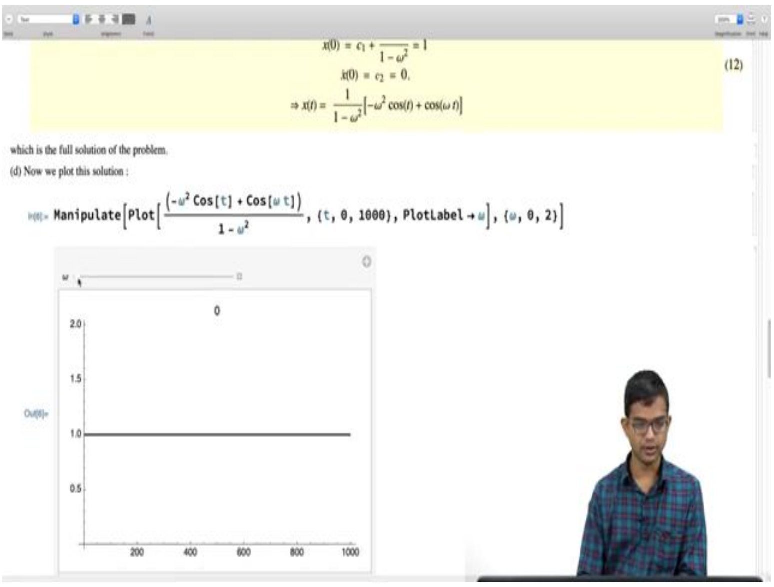

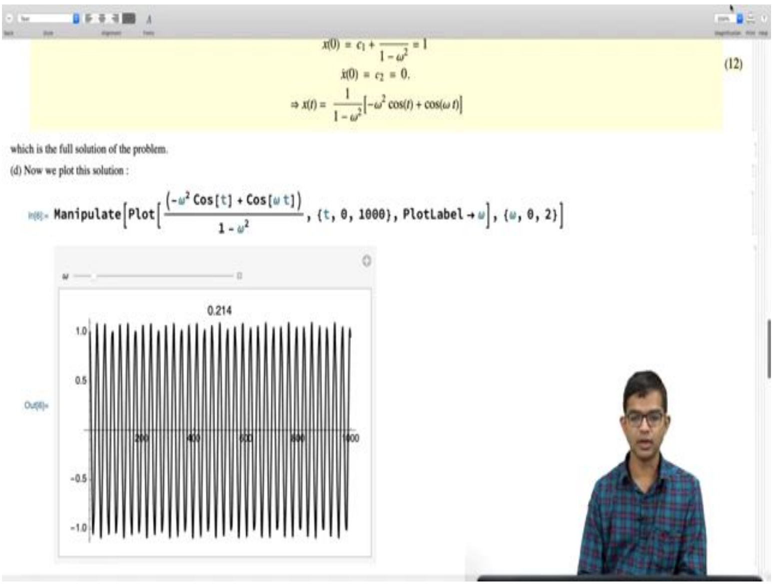

The complementary solution is, is well-known, right, it is simply $c1\ cos(t)\ +\ c2\ sin(t)$. So, this *c1* and *c2* are arbitrary constants. And they have to be fixed, based on initial conditions, right so this is where the initial conditions are hidden. The initial conditions are hidden in the complementary solution.

So, I told, I gave you the, particular solution has somehow washed away the initial condition, which is what you expect because a steady state must in fact lose all the information about where the system started from. So, in that sense, the particular solution better not carry any information about the initial conditions.

But here, *c1* and *c2* are 2 free constants, which we can determine based on the initial conditions. So, we will do that in a moment. Right, so this is a general solution. And the general solution of the full in-homogeneous differential equation, in-homogeneous in other words the forced differential equation. The driven oscillator problem is simply $c1\ cos(t)\ +\ c2\ sin(t) + 1/1-\omega^2\ \ cos(\omega\ t)$.

So, I told you it is as simple as that. You take the full general solution of the complementary solution. And then simply add it to the particular (sol) and you are done. This is the full general solution of the differential equation. And then we plug in the initial conditions.

We have told that $x(0)\ = 1$, $\dot{x}(0) = 0$. And this tells us that in our problem, we must fix our *c1* to be 1-($1/1-\omega^2$) and *c2* to be 0. And lo and behold, we have the full final answer.

This is the full solution for our particular problem. It has embedded in it, information about the initial conditions.

So, $x(t)$ is; now here I can go ahead and put $x(0)$, I can do $\dot{x}(0)$ and they will agree with the initial conditions that have been specified. Right so, now that we have the full solution, we can go ahead and plot it. Once again, I use this very nice comment called manipulate where I can vary this parameter omega and check out what it looks like.

So, I have when omega equal to 0, of course nothing is happening. And then I can slowly increase ω and then I see, I get a, you know, oscillatory solution, which is not a surprise looking at the kind of function, the form that I am plotting.

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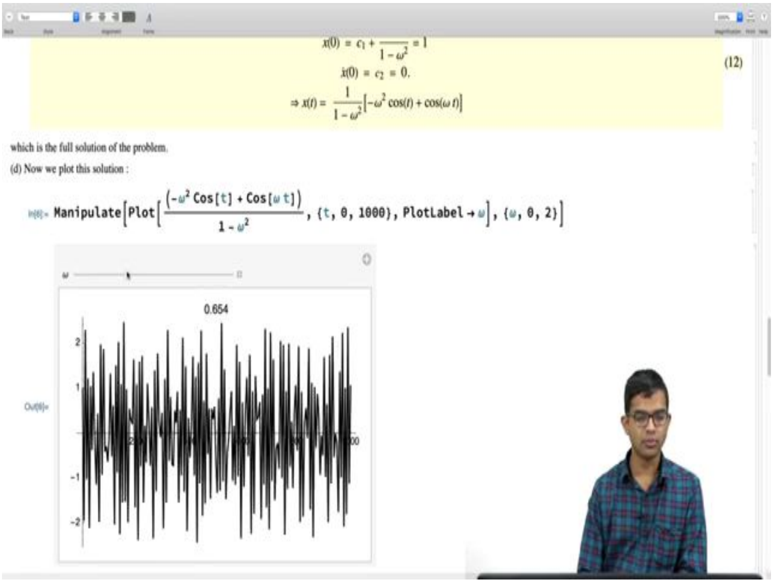
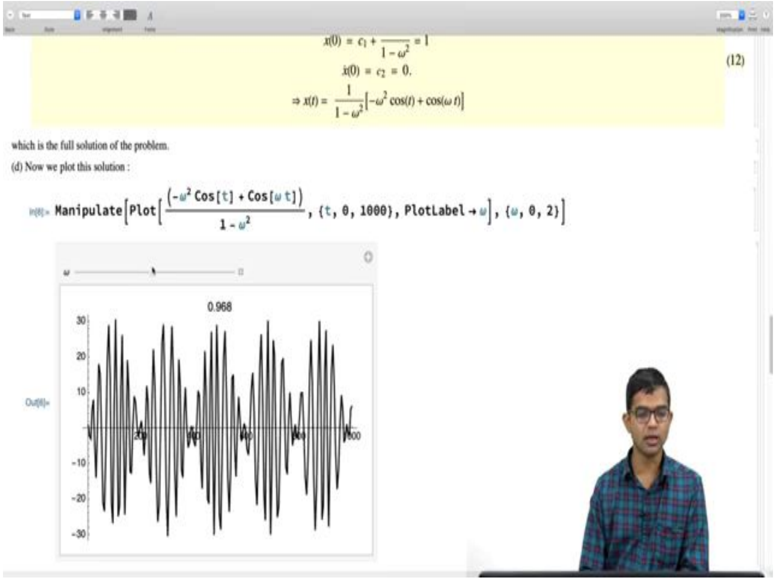
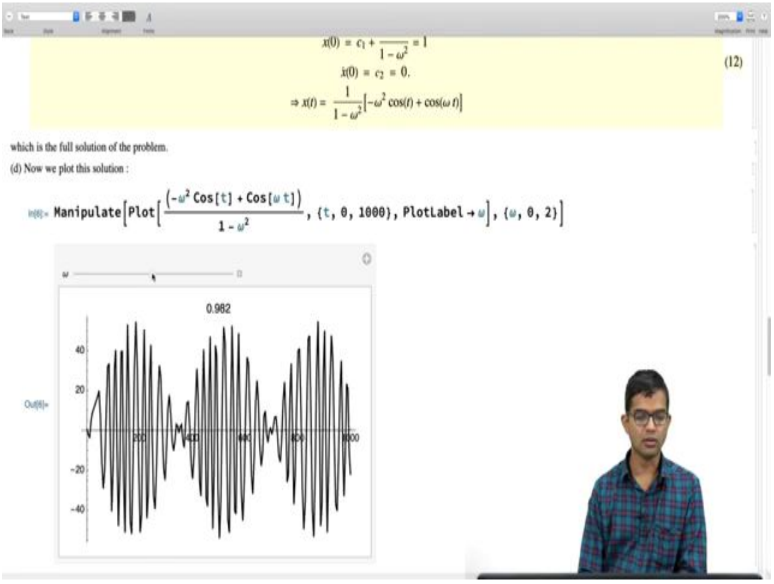




$\dot{x}(0) = c_2 = 0.$
(12)
$\Rightarrow x(t) = \frac{1}{1-\omega^2}[-\omega^2 \cos(t) + \cos(\omega t)]$
which is the full solution of the problem.
(d) Now we plot this solution :
In[6]:= Manipulate[Plot[(-ω^2 Cos[t] + Cos[ω t])/(1 - ω^2), {t, 0, 1000}, PlotLabel → ω], {ω, 0, 2}]
Out[6]=
0.968



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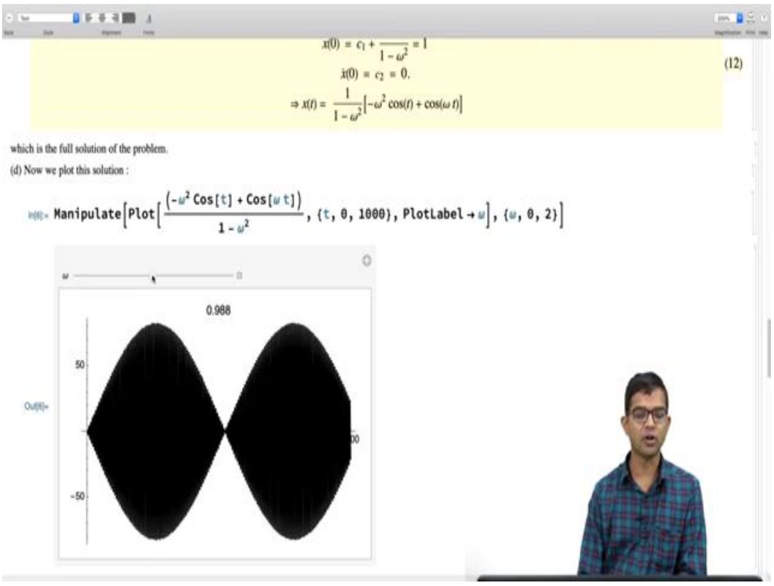
In[6]:= Manipulate[Plot[(-ω^2 Cos[t] + Cos[ω t])/(1 - ω^2), {t, 0, 1000}, PlotLabel → ω], {ω, 0, 2}]
Out[6]=
0.982



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In[6]:= Manipulate[Plot[(-ω^2 Cos[t] + Cos[ω t])/(1 - ω^2), {t, 0, 1000}, PlotLabel → ω], {ω, 0, 2}]
Out[6]=
0.988

So, what is very interesting is what happens as we go closer and closer to ω = 1, right, so I told you that the resonance happens at ω = 1. But if I am not sitting at ω, but very close to ω = 1, then I see these oscillations, but with a slightly different frequency. That is like a amplitude modulation happening here, right. So and if you think for a moment, you realize that this is really, all that is happening is some kind of beat phenomenon, right?

So, here we have I am trying to superpose 2 cosines. If both of these cosines; the frequencies corresponding to them are close to each other, that is when the beat phenomenon is evident. So, this is something that I urge you to go back and play with.

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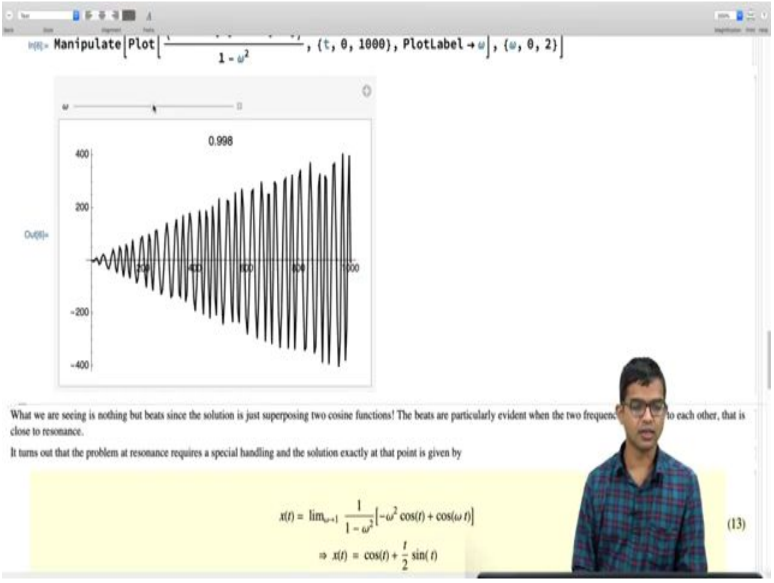

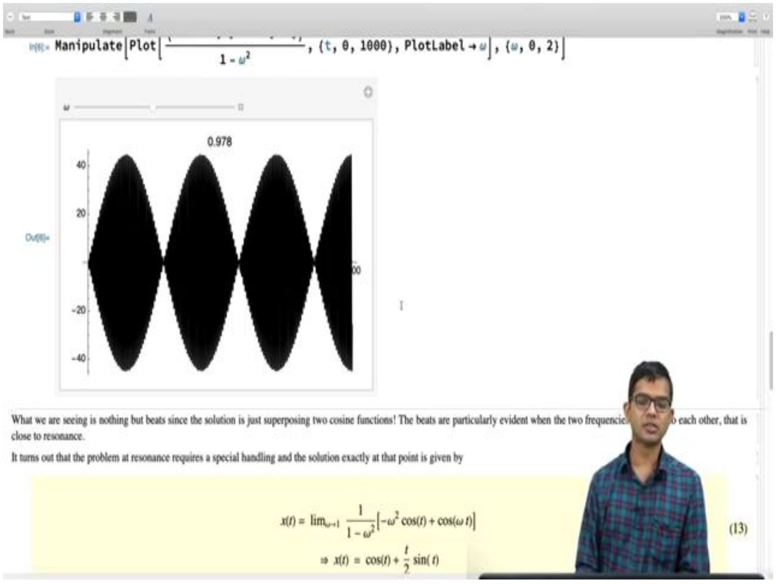

So, if you are sitting at one, of course it will not be seen. So, around 0.99 is when you see beat phenomenon in a very nice way. 0.99 is difficult to interpret, I guess. Okay so, this is something for you to play with. Right, so, this is a plot.

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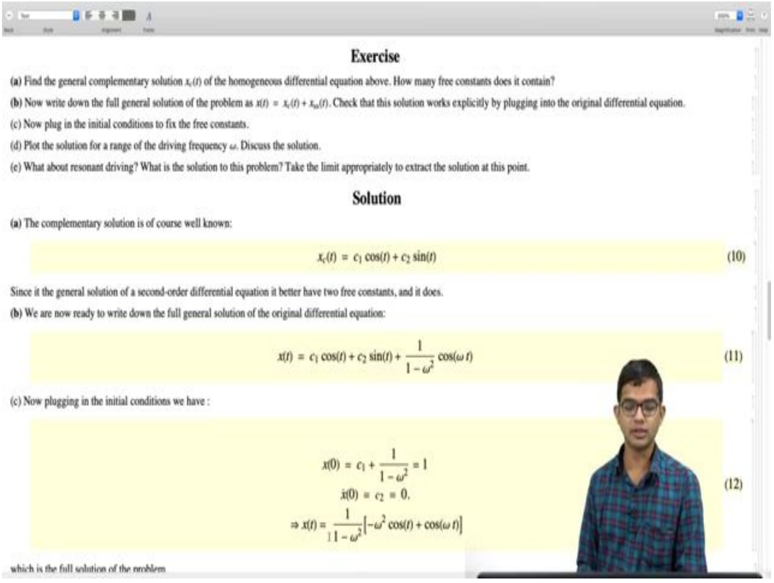

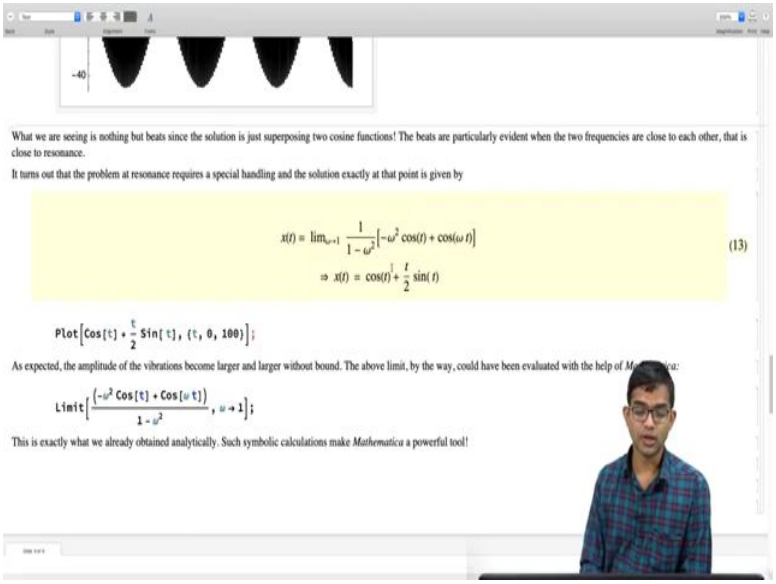

Now, let us ask the question. What happens when, if I put $\omega = 1$, right? After all, I have an external forcing term. I am free to choose this external forcing frequency, to be whatever I want.

So, but if it looks like if I just blindly put omega equal to 1 in this solution, it seems like it is going to blow up. So, why should I get a solution, which is blowing up, when the original differential equation is completely legitimate. There is no, if I had started my problem with the differential equation itself, if I had started with $\omega = 1$ and built in it, I should have been able to work out the full solution of it, which is ofcourse the case.

But it turns out that, what you have to do here is you cannot blindly just put $\omega = 1$, but you must carefully take the limit. So, it is of the 0/0 form and if you take the limit so, this is an exercise for you to do, right. I am not doing it here. But it is just a quick exercise, if you are familiar with this kind of stuff, which I assume you are.

You can use for example; the L'Hospital's rule. You take a derivative of the function in the numerator and derivative of the function in the denominator and then take the, put the limit ω going to 1.

Then you will get $x(t) = cos(t) + t/2\, sin(t)$, right, so this is, so a remark here is that, if you take a course on differential equation at some point, or if you might have already seen it, you will see that there are certain very special cases, where one has to be careful with the ansatz one makes.

And so, depending upon, depending upon the nature of the forcing term. So, you have $cos(\omega\ t)$. If ω is a certain value, which is connected to a certain route of a quadratic equation on the left-hand side, if it turns out to be equal to that one, right. I mean I am not going to give you the details, so that you can look up some textbook on differential equations.

Then the correct ansatz will be actually $t\, sin(t)$. So, this factor of t would come in, even within the formal theory of differential equations itself. So, that way you could have got to this solution directly. If you had started your original differential equation itself, put $\omega = 1$ and worked it out, then you would have had to you know implant this $t\, sin(t)$ itself.

So, this is an exercise for you to try and carry out. But if you missed this remark, or if you did not follow this remark, maybe you should just wait till you encounter the theory of differential equations in some slightly more advanced course. But for now, let us say that just taking the limit $\omega = 1$ in the right way, will already give us this solution. So, let us plot this.

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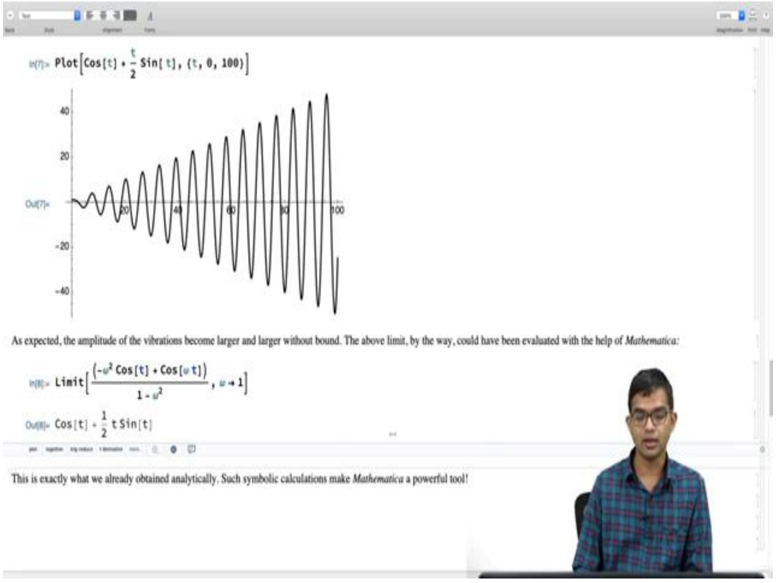

Plotting is very instructive. So, we see that, as for larger and larger time, the amplitude keeps on increasing; right. So, there is a very systematic way, in which the amplitude will keep increasing. And if I take the limit t going to infinity, then I have these oscillations become really large. And, so this is something that we have already seen, right.

Where we saw, we saw that the square of the amplitude of the steady state solution is infinite at $\omega = 1$. So, we are considering this $\omega = 1$ case. But we are looking at the full solution; $x(t)$ as a function of time is completely well-defined, because this is the, this includes the transients, right. So, there is a transient part and there is a part which is called the steady state.

So and at any finite time, there is a very precise value of $x(t)$, right. It is just that, you see from this plot, that for larger and larger times, the amplitude of oscillations becomes very-very large, so this is quite instructive. So, by the way, you could have also used or we could have taken Mathematica's help to evaluate this limit. So, there is this function called limit.

You can just plug in the function there and take the limit ω going to 1, it will give you $cos(t) + 1/2\, t\, sin(t)$, which is what we obtained analytically, right so Mathematica is a powerful tool to cross check your results.

Sometimes, sometimes it is good to have multiple ways of checking, right, so it is a, even though Mathematica can directly do it for you, it is better to use it as a cross checking device.

First, do it yourself and then get Mathematica to do it, cross check and then depend entirely on Mathematica only for problems, where you cannot do it. Like for example, hardcore numeric, so that is why we should do a lot of checks for small cases, simple cases and then stretch it to the more difficult cases later on. So, it is a standard tool.

So, now what we want to do? We want to see, if we can reproduce this from a purely numerical method right, which is in the spirit of this whole course, which is in the spirit, or the philosophy behind this course is to find a numerical perspective for this.

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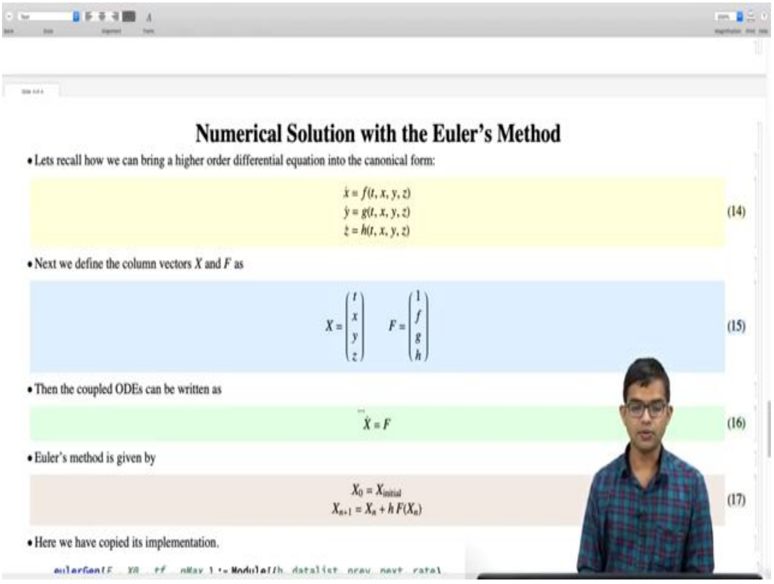

So, let us recall how we can take any higher order differential equation and put it into the canonical form. Right so, I told you that this differential equation that we have is linear, but it is of order 2, second order linear differential equation, inhomogeneous differential equation. That is the classification from a differential equation point-of-view. So, no matter what the order of your differential equation is, it is possible to bring it down to effectively first order.

But involving more variables and that is the form that is most suitable for the application of a numerical method, right like the Euler's method is the simplest numerical way of solving a differential equation. If you have only one variable, all it entails is, saying okay, if I want to solve dx/dt is equal to some function x, t or I will replace dx/dt by $x_{n+1} - x_n/\Delta$.

So, $x_{n+1} = x_n + \Delta$ times, the derivative at that point right, numerical. And then you go to the next step. And then you go to the next step and you slowly build it up. And there is a small delta, right, which controls the accuracy of your numerical method. The smaller it is the more accurate it is. And if you want a greater accuracy, you will have to shrink your delta and therefore, it will consume more resources.

So, later on we will see how you can dramatically improve the efficiency of this method, by doing some small modification. So, there is something called the improved Euler method. And then there is a more fancy one, which is called the Runge-Kutta method. So, improved Euler method itself is a, is a form of the RK methods.

So, we will discuss these at a later time. But for now, let us recall how to recast our higher order differential equation into the canonical form. So, what you do is you just simply introduce more variables. So, you had $\ddot{x}$. So but you define $\dot{x}$ itself as *y*. So, in this case, let us do this.

So, I have, we will see that example in a moment, right so, if you have $\dot{x} = F$, $\dot{y} = G$ and $\dot{z} = H$, you can rewrite your whole equation as the derivative of a vector $\dot{x} = F$. And that Euler's method simply gives you $x_0 = x_{\text{initial}}$ and $x_{n+1} = x_n + H * F(x_n)$, which is what I just said. And which will be discussed in a separate video in detail, right.

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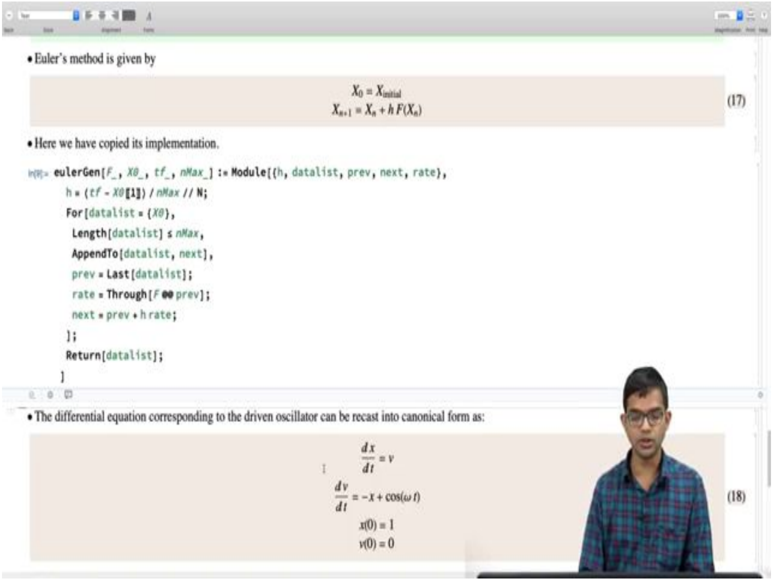

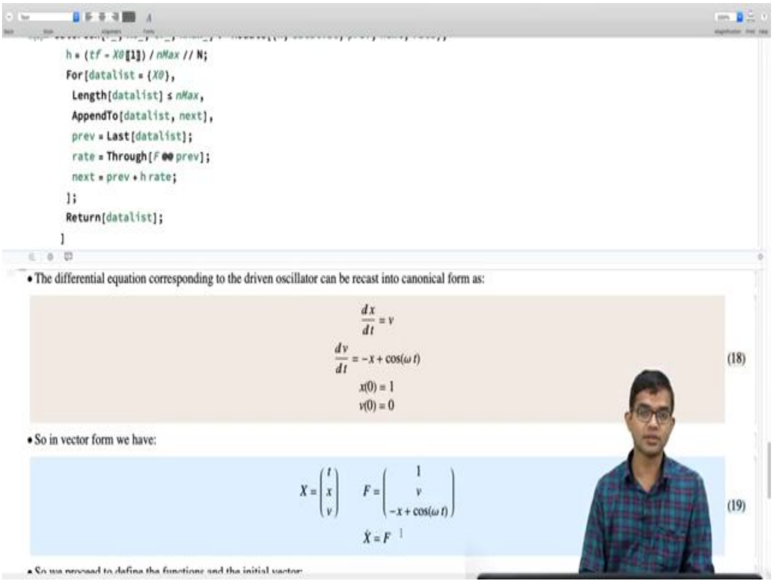

So, we have this implementation of the Euler general method, which was carried out in an earlier video, which you should go back and watch. So, I am not going to go into the details of this, right so, this is basically implementing this algorithm. So, there is a nice way of, a compact way of putting all these together in inputs and then here I have got, I have just borrowed this code.

I am going to run this and then use it. So, first I have to, even if I have to use this as a black box, I should know how to input stuff, how to output stuff, how to analyze it, right? So, my differential equation is d^2x/dt^2 plus bla-bla-bla. So, first I am going to introduce the new variable $dx/dt = v$. Then I have $dv/dt = -x + cos(\omega t)$. So, that is enough.

So, I have only 2 variables. And then I have $x(0) = 1$, $v(0) = 0$. So, in vector form, I have capital X = t, is also defined as a variable here, because it is convenient to do numerics in this way. Then my differential equation can be just simply written down as $\dot{X} = F$. It is a effectively first order differential equation, but for a vector now. Then I have to define my identity, right.

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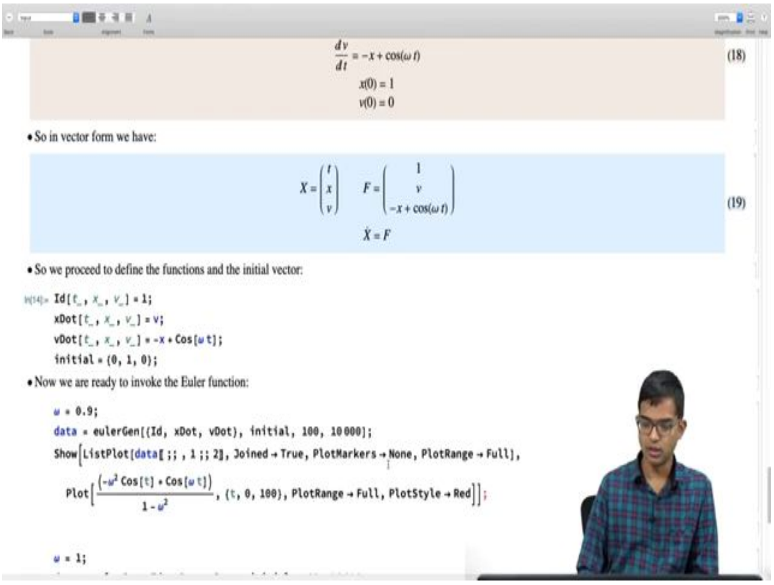

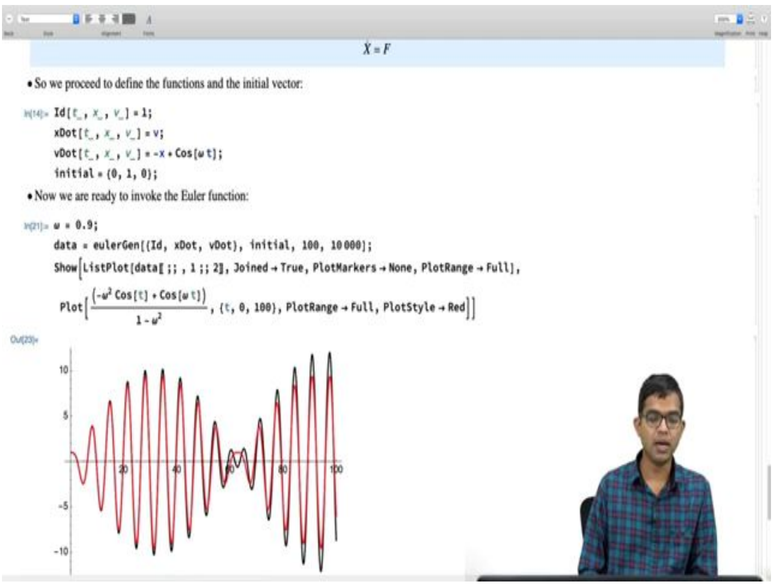

So, I will define an identity function. So, it takes in the input of time, position and speed t, x and v. But it just gives me 1, because I know that $\dot{X} = F$ if I look at the first row, I just want it to be 1.

To get the second row, I use $\dot{x} = v$. So, all of this I implement in one cell. There you go. And then I also have to define my initial vector. So, I know that at time $t = 0$, my position is 1 and speed is 0. And then if I, if I choose a certain omega and then I can simply go ahead and implement this. So, I have to put in the, these functions Is $\dot{x}$ and $\dot{v}$.

Then I have to give you the initial condition, which I have already written down here 010. Then I have to choose these numbers 100 and 10 000. So, 100 is gives me the time steps, up to which I am going to run my simulation, up to a certain time t. And that is chosen to be 100.

And 10 000 is, is where the information about the individual time step. So, the larger this number here is, the more fine the simulation is. And so the more accurate it is going to be, right. So of course the flip side is that it is going to take longer for your code to run. So, let me run this for 10 000. So, I have the data has come out all the way. And I am going to compare against the analytical solution I have, there you go.

So, you see that the agreement is good but around these bends, it is somewhat, it misses the details, right so this has to do with the fact that, you know whenever there are these kind of bends, the derivative is inaccurate. And the inaccuracy in the derivative is exaggerated. And therefore, the agreement between the numerics and the theory is not that great. So, you can go ahead and play with this. So, maybe I will try one more.

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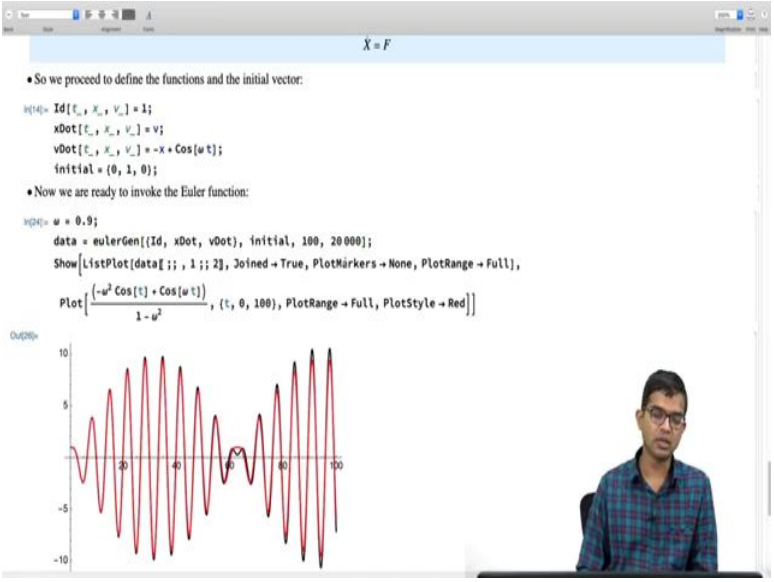

If I make this 20000 let us see if the accuracy becomes better. It has become slightly better. But, but if you want to really increase it to a much larger time, it is going to take a while for it to run. So, I will allow you to play with this. But a better thing to do is, actually to improve the quality of your algorithm, right.

So, you can use something called the Improved Euler function method, which we will discuss at some point or it is going to be part of our home-work. And even better than Improved Euler is the RK 4 method, which is also something that we will discuss after some time. So, let us see what happens, if I put omega equal to 1.

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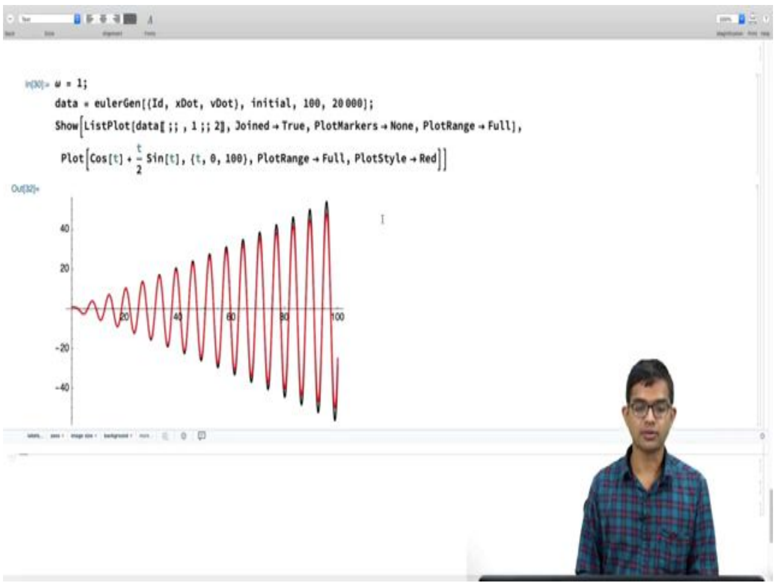

So, if I put $\omega = 1$, it is going to take a while. So, there you go. Once again, we see that surely there is very good agreement. But it is not, it is not exact, the agreement between the analytical curve and the purely numerical run.

And it is going to miss by a larger and larger margin, as time increases, particularly around these, the point of highest amplitude and point of lowest amplitude, right. Whenever the system is turning around sharply, that is where the information about the derivative is not so accurate.

Okay so, that is what I wanted to cover in this module. So, the main message here is that there is the physics of the driven oscillator comes out very nicely. So, a full analytical solution is possible. And with the help of Mathematica or with the help of a numerical tool, like the Euler Method, which is a very-very simple way to solve a differential equation. And we show that it is possible to solve it numerically and the 2 approaches give us results, which agree very closely.

And if you want to make the agreement better and better, you must choose your discretization. You must make, you know the runs in your numerical program, the times involved for, you know the incremental time involved should become smaller and smaller.

Right so, there are details of how the error scales with H and so on is not the focus of this course. But if you take a course on numerical methods or you know computation methods and so on, there is a very systematic and mathematical way to work out the strength or weakness of an algorithm quantitatively.

You can say why Euler method is you know the errors are of a certain order and how Improved Euler is better and how RK4 is even better and so on. But let us not go into that, that is not the focus of this course, at least at this point. The main message here is of course these 2 methods work and we should go back and play more.

And so, the one thing that you should try out is, to work out the same problem for the circuit equivalent. And the other thing is to include a damping term. So, you know the same kind of techniques will hold, you can have a damping term where, you know there is a frictional force, which tends to slow down your particle.

The greater its speed, the smaller its, so the greater is the frictional force. So, you can have a term like $d^2x/dt^2 + x$ but also something like dx/dt with a constant $v - b\, dx/dt$ in the original equation. So, that is also worth trying out. But that is it for now. We will come back with more improved methods in the next morning. Thank you.